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Propagation of electroacoustic waves in the transversely isotropic piezoelectric medium reinforced by randomly distributed cylindrical inhomogeneities

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Abstract

The propagation of electroacoustic waves in a piezoelectric medium containing a statistical ensemble of cylindrical fibers is considered. Both the matrix and the fibers consist of piezoelectric transversely isotropic material with symmetry axis parallel to the fiber axes. Special emphasis is given on the propagation of an electroacoustic axial shear wave polarized parallel to the axis of symmetry propagating in the direction normal to the fiber axis.

The scattering problem of *one isolated continuous fiber* ("one-particle scattering problem") is considered. By means of a Green's function approach a system of coupled integral equations for the electroelastic field in the medium containing a single inhomogeneity (fiber) is solved in closed form in the long-wave approximation. The total scattering cross-section of this problem is obtained in closed form and is in accordance with the electroacoustic analogue of the *optical theorem*.

The solution of the one-particle scattering problem is used to solve the homogenization problem for a *random set* of fibers by means of the self-consistent scheme of *effective field method*. Closed form expressions for the dynamic characteristics such as total cross-section, effective wave velocity and attenuation factor are obtained in the long-wave approximation.

Keywords: Electroacoustic waves; Scattering; Dynamic electroelastic inhomogeneity; Dynamic electroelastic Green's function; Self consistent scheme; Effective field method; Dynamic characteristics of piezoelectric fiber-composites

1. Introduction

Piezoelectric composites are an important branch of modern engineering materials with wide applications in actuators and sensors in "smart" materials and structures. Combining two or more distinct

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constituents, piezoelectric composite materials can take the advantages of each constituent and have superior electromechanical coupling characteristics compared to homogeneous piezoelectrical materials. These materials have been developed in many forms including second phase piezoelectric inclusions embedded in a polymer matrix and polymer filled piezoelectric inclusions in a solid piezoelectric ceramic matrix. The secondary-phase piezoelectric inclusions in the matrix of composites can be continuous fibers, short fibers, holes, voids or dispersed quasispherical particles. A further important application of fiber reinforced piezocomposites is in the health monitoring of structures (Lin and Chang, 1999). More recently, piezoelectric composites are extensively used as transducers for sonar projectors and for ultrasonic applications (Tressler and Uchino, 2000).

Among various types of piezocomposites, the fiber reinforced composites consisting of a set of parallel continuous cylindrical rods of piezoelectric ceramic in a matrix were identified as most promising for ultrasonics (Tressler and Uchino, 2000). For example, composite sensors containing piezoelectric ceramic rods in a polymer-based matrix are widely used in trandusers for underwater and biomechanical imaging applications (Gururaja et al., 1981).

Recent developments of the micromechanical modeling of piezoelectric composites have been carried out by many researchers. The rule of mixtures (one of the simplest schemes in the mechanics of composites) was applied to fiber reinforced piezocomposites (Chan and Unsworth, 1989). This rule, however, may not be fully reliable. Indeed, even in the case of purely elastic properties the rule of mixture, while accurately predicting the effective stiffness along the fiber direction, may yield errors in other electroelastic constants if the contrast in properties between the matrix and the fibers is substantial—such a limitation is relevant for the ceramic–polymer composites.

Grekov et al. (1989) used a model of coaxial cylinders placed in a matrix with effective properties to estimate the properties of a piezoelectric matrix reinforced by piezoelectric fibers. Their calculation, however, covers only three (out of ten) effective constants.

Smith and Auld (1991) and Smith (1993) analyzed the effective properties of fiber reinforced piezocomposites using the following assumptions: (a) in the direction along the fiber, matrix and fibers share the same strain; (b) in the plane normal to fibers, the matrix and the fibers carry the same stresses and (c) electrical field in the plane normal to fibers and all the shear strains are assumed to be zero. On the basis of these assumptions, the authors obtain six (out of ten) effective constants. We note that, whereas the assumption (a) is fully justified, the assumption (b) is less solid: in the mechanics of composites such assumption has been shown to be generally inaccurate (Hill, 1963). Note also that the mentioned work cannot be readily extended to the case when the matrix is piezoelectric.

Getman and Mol'kov (1992) considered a periodic arrangement of piezoelectric fibers in a piezoelectric matrix. Their results, however, were not given in the closed form and were illustrated only for the case of fibers with zero stiffness, conductivities and piezoelectric constants (porous piezoceramic).

Wang (1992) considered the problem of the piezoelectric material reinforced by piezoelectric fibers and calculated the effective constants in the non-interaction approximation (low concentration of fibers). His calculations cover seven (out of ten) effective constants.

Chen (1993) considered a fibrous piezocomposite in the very special case when the shear moduli of the matrix and of the fibers coincide (this case may not be relevant for real piezocomposites). In this special case, extending the ideas of Hill (1964) on the effective elastic moduli of a two phase composite, he derived closed form expressions for the effective constants. The same problem was considered by Chen (1994), without the assumption of equal shear moduli, in the framework of Mori-Tanaka's method, but the set of calculated effective constants was incomplete (seven out of ten).

The method of effective field (discussed below) was applied (Levin, 1996a, 1999; Levin et al., 1999) to the problem of effective properties of an isotropic purely elastic (piezopassive) matrix with spheroidal piezoelectric inhomogeneities of identical aspect ratios that are either parallel or randomly oriented (note that in the latter case the piezoeffect is lost at the macroscale). By Levin (1996a) the transversely isotropic matrix reinforced by the continuous fibers was also considered. This self-consistent scheme (effective field method) leads to explicit expressions for all ten effective piezoelectric constants, the dependence on the volume fraction of fibers remains physically reasonable in all range of fraction changing (from 0 to 1).

All the results mentioned above are achieved in the framework of statics. It is well known (Kunin, 1983) that even if the components of the original heterogeneous material are purely elastic, dynamic processes will cause an effective medium with attenuation and dispersion because of wave scattering on the inclusions and the existence in such a medium of intrinsic length parameters. All these dynamic characteristics cannot be described appropriately in a static framework. Due to the increasing need of an understanding of dynamic processes in piezoelectric composites, it is highly desirable to establish modelling of effective characteristics in a fully *dynamic* framework. It is the goal of this paper to study some effective *dynamic material characteristics* of a fiber reinforced piezocomposite.

The paper is organized as follows: In Section 2 we derive from the equation of motion and charge conservation law integral equations for the scattered electroelastic fields for an *isolated* inhomogeneity. The solution of this problem is crucial for solving the scattering problem of a statistical ensemble of randomly distributed inhomogeneities. In Section 3 we consider a transversely isotropic medium containing one isolated continuous fiber consisting of transversely isotropic piezoelectric material with different moduli from the matrix but with coincident symmetry axis with the fiber axis. Due to the quasiplane symmetry of the problem we introduce the electroelastic quasiplane dynamic Green's function which is used to formulate a set of integral equations for the scattered electroelastic fields of an isolated continuous fiber. A consideration of dynamic Green's functions for piezoelectric, thermoelastic and poroelastic infinite three dimensional media can be found, e.g. Norris (1994).

The "one-particle" scattering problem is solved for the case when the diameter of fiber is much smaller than the wavelength of incident electroelastic field (long-wave-approximation). In this regime we solve the integral equations and give closed form expressions for the scattered electroacoustic fields. In Section 4 we derive for the case of an isolated fiber the total cross-section by utilizing the electroacoustic analogue of *optical theorem*. To that end the farfield asymptotics of the scattered electroacoustic fields are derived. In Section 5 the propagation of an axial shear wave on a *random set* of continuous fibers having identical radii and parallel axes of symmetry is considered. The solution of the "multiple-particle" scattering problem is formulated in the framework of a self-consistent scheme of integral equations (effective field method). By introducing statistical hypotheses on the distribution of the fibers the multiple scattering problem is reduced to an effective "one-particle" scattering problem. In the framework of this approach, effective electroacoustic fields and the dynamic electroelastic characteristics are calculated explicitly. Finally in Section 6 the effective wave velocity and attenuation factor are calculated in explicit form.

2. Integral equations for the scattering problem

We consider a piezoelectric medium obeying the following linear constitutive equations

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - e_{kij}E_k \\ D_i &= e_{ikl}\varepsilon_{kl} + \eta_{ik}E_k \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain tensors, $\mathbf{E}$ and $\mathbf{D}$ are the electric field intensity and electric displacement respectively, $\boldsymbol{C} = \boldsymbol{C}^{\mathrm{E}}$ is the tensor of elastic moduli at fixed $\mathbf{E}$, $\boldsymbol{\eta} = \boldsymbol{\eta}^{\varepsilon}$ is the permittivity tensor at fixed strain $\boldsymbol{\varepsilon}$, $\boldsymbol{e}$ is the piezoelectric constants tensor.

The substitution of relations (1) into the equations of elastodynamics and Maxwell's equations leads to a coupled system of equations of linear electroelasticity. As usual, we disregard body forces of electrical nature. Hence, the equations of motion have the same form as in the theory of elasticity

$$\partial_j \sigma_{ij} - \rho \ddot{u}_i = -Q_i, \quad \partial_j = \partial/\partial x_j \tag{2}$$

where u_i is the vector of elastic displacement, ρ is the material's density, Q_i is the body force vector.

The solution of Eq. (2) together with Maxwell's equations describes the elastic–electromagnetic waves, i.e. elastic waves interacting with the electric field and the electromagnetic waves accompanying the deformation. If the characteristic velocity of the elastic waves is v, then the corresponding velocity of the electromagnetic waves has the order of $10^5 v$. Therefore, we neglect the magnetic field generated by the elastic field propagating with velocity v. It follows, then, that the magnetic effects can be neglected and the quasistatic approximation for the electric field can be used.

An additional field equation is the conservation of free electric charges:

$$\partial_i D_i = -q \tag{3}$$

where q is the density of free electric charges and D_i is the electric displacement. Since

$$E_i = -\partial_i \varphi, \quad \varepsilon_{ij} = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i) \tag{4}$$

where φ is the electric potential, the constitutive equations can be rewritten in the form

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\partial_l u_k + e_{kij}\partial_k \varphi \\ D_i &= e_{ikl}\partial_l u_k - \eta_{ik}\partial_k \varphi \end{aligned} \tag{5}$$

Substituting them into (2) and (3) yields a coupled system of linear differential equations of electroelasticity for the piezoelectric medium:

$$\begin{aligned} \partial_j C_{ijkl}\partial_l u_k + \partial_j e_{kij}\partial_k \varphi - \rho \ddot{u}_i &= -Q_i \\ \partial_i e_{ikl}\partial_l u_k - \partial_i \eta_{ik}\partial_k \varphi &= -q \end{aligned} \tag{6}$$

We consider now an harmonic oscillation of the medium with frequency ω. Since the dependence of quantities entering (6) on time is given by multiplier $\exp(-\mathrm{i}\omega t)$, the system (6) takes the form

$$\begin{aligned} \partial_j C_{ijkl}\partial_k u_l + \rho\omega^2 u_i + \partial_j e_{jik}\partial_k \varphi &= -Q_i \\ \partial_i e_{ikl}\partial_k u_l - \partial_i \eta_{ik}\partial_k \varphi &= -q \end{aligned} \tag{7}$$

Let the density of the body forces Q_i and electric charges q be localized within a domain V. The solution of the system (7) that vanishes at infinity can be represented as

$$\begin{aligned} u_i(\mathbf{x}) &= \int_V G_{ik}(\mathbf{x}-\mathbf{x}')Q_k(\mathbf{x}')\,\mathrm{d}\mathbf{x}' + \int_V \Gamma_i(\mathbf{x}-\mathbf{x}')q(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \\ \varphi(\mathbf{x}) &= \int_V \gamma_k(\mathbf{x}-\mathbf{x}')Q_k(\mathbf{x}')\,\mathrm{d}\mathbf{x}' + \int_V g(\mathbf{x}-\mathbf{x}')q(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \end{aligned} \tag{8}$$

(the dependencies on frequency ω are omitted). The substitution of these expressions into the left-hand parts of (7) leads to a system of differential equations for the kernels $G_{ik}(\mathbf{x})$, $\Gamma_i(\mathbf{x})$, $\gamma_k(\mathbf{x})$ and $g(\mathbf{x})$—the components of the electroelastic Green's function:

$$\begin{aligned} &(C_{ijkl}\partial_j\partial_k + \rho\omega^2\delta_{il})G_{lm}(\mathbf{x}) + e_{jik}\partial_j\partial_k\gamma_m(\mathbf{x}) = -\delta_{im}\delta(\mathbf{x}) \\ &(C_{ijkl}\partial_j\partial_k + \rho\omega^2\delta_{il})\Gamma_l(\mathbf{x}) + e_{jik}\partial_j\partial_k g(\mathbf{x}) = 0 \\ &e^{\mathrm{T}}_{ikl}\partial_i\partial_k G_{lm}(\mathbf{x}) - \eta_{ik}\partial_i\partial_k\gamma_m(\mathbf{x}) = 0 \\ &e^{\mathrm{T}}_{ikl}\partial_i\partial_k \Gamma_l(\mathbf{x}) - \eta_{ik}\partial_i\partial_k g(\mathbf{x}) = -\delta(\mathbf{x}) \end{aligned} \tag{9}$$

where $\delta(\mathbf{x})$ is the spatial Dirac's δ-function. Fourier transformation of these equations yields

$$
\begin{aligned}
&\Lambda_{il}(k)G_{lj}(\mathbf{k}) + h_i(\mathbf{k})\gamma_j(\mathbf{k}) = \delta_{ij} \\
&h_l^{\mathrm{T}}(\mathbf{k})G_{lj}(\mathbf{k}) - \lambda(\mathbf{k})\gamma_j(\mathbf{k}) = 0 \\
&\Lambda_{il}(\mathbf{k})\Gamma_l(\mathbf{k}) + h_i(\mathbf{k})g(\mathbf{k}) = 0 \\
&h_l^{\mathrm{T}}(\mathbf{k})\Gamma_l(\mathbf{k}) - \lambda(\mathbf{k})g(\mathbf{k}) = 1
\end{aligned} \tag{10}
$$

where

$$
\Lambda_{il}(\mathbf{k},\omega) = k_j C_{ijkl} k_k - \rho\omega^2\delta_{il}, \quad h_i(\mathbf{k}) = e_{kil}k_k k_l, \quad h_l^{\mathrm{T}}(\mathbf{k}) = e_{ikl}^{\mathrm{T}} k_i k_k, \quad \lambda(\mathbf{k}) = \eta_{ik}k_i k_k \tag{11}
$$

The solution of the system (10) can be written in the form

$$
\begin{aligned}
&G_{ik} = \left(\Lambda_{ik} + \frac{1}{\lambda}h_i h_k^{\mathrm{T}}\right)^{-1}, \quad g = -(\lambda + h_i^{\mathrm{T}}\Lambda_{ij}^{-1}h_j)^{-1} \\
&\gamma_i = \frac{1}{\lambda}h_k^{\mathrm{T}}G_{ki}, \quad \Gamma_i = -\Lambda_{ik}^{-1}h_k g
\end{aligned} \tag{12}
$$

The symmetry of system (10) indicates that $\gamma_i = \Gamma_i$. Introducing the notation

$$
\mathbf{G}(\mathbf{k},\omega) = \begin{pmatrix} G_{ik}(\mathbf{k},\omega) & \gamma_i(\mathbf{k},\omega) \\ \gamma_k^{\mathrm{T}}(\mathbf{k},\omega) & g(\mathbf{k},\omega) \end{pmatrix} \tag{13}
$$

The $\mathbf{x}$-representation of Green's function can be obtained via the inverse Fourier transformation:

$$
\mathbf{G}(\mathbf{x},\omega) = \frac{1}{(2\pi)^3}\int \mathbf{G}(\mathbf{k},\omega)\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{k} \tag{14}
$$

Let us consider now an unbounded piezoelectric medium with electroelastic characteristics $\mathbf{L}^0$, containing the region V (inclusion) with different electroelastic properties $\mathbf{L}$. We start with the following system of differential equations for the electroelastic fields in such a medium

$$
\begin{aligned}
&\partial_j C_{ijkl}(\mathbf{x})\partial_l u_k(\mathbf{x}) + \rho(\mathbf{x})\omega^2 u_i(\mathbf{x}) + \partial_j e_{jik}(\mathbf{x})\partial_k\varphi(\mathbf{x}) = 0 \\
&\partial_i e_{ikl}(\mathbf{x})\partial_l u_k(\mathbf{x}) - \partial_i\eta_{ik}(\mathbf{x})\partial_k\varphi(\mathbf{x}) = 0
\end{aligned} \tag{15}
$$

Here $C(\mathbf{x})$, $e(x)$, $\eta(\mathbf{x})$ and $\rho(\mathbf{x})$ are functions of coordinates which are equal to $\boldsymbol{C}^0$, $\boldsymbol{e}^0$, $\boldsymbol{\eta}^0$, ρ^0 in the main material (matrix) and $\boldsymbol{C}$, $\boldsymbol{e}$, $\boldsymbol{\eta}$, ρ inside of inclusion. One may represent functions $\boldsymbol{C}(\mathbf{x})$, $\boldsymbol{e}(\mathbf{x})$, $\boldsymbol{\eta}(\mathbf{x})$, $\rho(\mathbf{x})$ as the sums

$$
\begin{aligned}
&\boldsymbol{C}(\mathbf{x}) = \boldsymbol{C}^0 + \boldsymbol{C}^1(\mathbf{x}), \quad \boldsymbol{e}(\mathbf{x}) = \boldsymbol{e}^0 + \boldsymbol{e}^1(\mathbf{x}), \quad \boldsymbol{\eta}(\mathbf{x}) = \boldsymbol{\eta}^0 + \boldsymbol{\eta}^1(\mathbf{x}), \quad \rho(\mathbf{x}) = \rho_0 + \rho_1(\mathbf{x}) \\
&\boldsymbol{C}^1(\mathbf{x}) = \boldsymbol{C}^1 V(\mathbf{x}), \quad \boldsymbol{e}^1(\mathbf{x}) = \boldsymbol{e}^1 V(\mathbf{x}), \quad \boldsymbol{\eta}^1(\mathbf{x}) = \boldsymbol{\eta}^1 V(\mathbf{x}), \quad \rho_1(\mathbf{x}) = \rho_1 V(\mathbf{x})
\end{aligned} \tag{16}
$$

where $V(\mathbf{x})$ is the characteristic function of the region V occupied by the inclusion and the quantities with the superscript "1" denote the differences

$$
\boldsymbol{C}^1 = \boldsymbol{C} - \boldsymbol{C}^0, \quad \boldsymbol{e}^1 = \boldsymbol{e} - \boldsymbol{e}^0, \quad \boldsymbol{\eta}^1 = \boldsymbol{\eta} - \boldsymbol{\eta}^0, \quad \rho_1 = \rho - \rho_0 \tag{17}
$$

Representation (16) allows to rewrite the system (15) in the form

$$
\begin{aligned}
&\partial_j C_{ijkl}^0\partial_l u_k(\mathbf{x}) + \rho_0\omega^2 u_i(\mathbf{x}) + \partial_j e_{jik}^0\partial_k\varphi(\mathbf{x}) = -\left[\partial_j C_{ijkl}^1(\mathbf{x})\partial_l u_k(\mathbf{x}) + \rho_1(\mathbf{x})\omega^2 u_i(\mathbf{x}) + \partial_j e_{jik}^1(\mathbf{x})\partial_k\varphi(\mathbf{x})\right] \\
&\partial_i e_{ikl}^{T0}\partial_l u_k(\mathbf{x}) - \partial_i\eta_{ik}^0(\mathbf{x})\partial_k\varphi(\mathbf{x}) = -\left[\partial_i e_{ikl}^{1T}(\mathbf{x})\partial_l u_k(\mathbf{x}) - \partial_i\eta_{ik}^1(\mathbf{x})\partial_k\varphi(\mathbf{x})\right]
\end{aligned} \tag{18}
$$

The right-hand side of Eqs. (18) can be considered as a distribution of body forces and electric charges, and one can replace these equations by a system of integral equations. This system can be written in the following short form

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}^0(\mathbf{x}) + \int_V \mathbf{S}(\mathbf{x}-\mathbf{x}')\mathbf{L}^1\boldsymbol{F}(\mathbf{x}')\,\mathrm{d}\mathbf{x}' + \omega^2\rho_1\int_V \mathbf{G}(\mathbf{x}-\mathbf{x}')\mathbf{J}\mathbf{f}(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \tag{19}$$

where it is denoted

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} u_i(\mathbf{x}) \\ \varphi(\mathbf{x}) \end{pmatrix}, \quad \mathbf{S}(\mathbf{x}) = \begin{pmatrix} G_{ik,l}(\mathbf{x}) & \gamma_{i,k}(\mathbf{x}) \\ \gamma^{\mathrm{T}}_{k,l}(\mathbf{x}) & g_{,k}(\mathbf{x}) \end{pmatrix}, \quad \mathbf{L}^1 = \begin{pmatrix} \boldsymbol{C}^1 & \boldsymbol{e}^1 \\ \boldsymbol{e}^{\mathrm{T}1} & -\boldsymbol{\eta}^1 \end{pmatrix},$$
$$\boldsymbol{F}(\mathbf{x}) = \begin{pmatrix} \varepsilon_{ij}(\mathbf{x}) \\ -E_i(\mathbf{x}) \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} \delta_{ik} & 0 \\ 0 & 0 \end{pmatrix} \tag{20}$$

with $\mathbf{f}^0(\mathbf{x}) = (u_i^0, \varphi^0)$ denoting the "incident" fields and superscript "T" denotes the transposed tensor. The incident fields satisfy the equations

$$\begin{aligned} &\partial_j C^0_{ijkl}\partial_l u^0_k(\mathbf{x}) + \rho_0\omega^2 u^0_i(\mathbf{x}) + \partial_j e^0_{jik}\partial_k\varphi^0(\mathbf{x}) = 0 \\ &\partial_i e^{T0}_{ikl}\partial_l u^0_k(\mathbf{x}) - \partial_i\eta^0_{ik}(\mathbf{x})\partial_k\varphi^0(\mathbf{x}) = 0 \end{aligned} \tag{21}$$

It follows from (19) that the strain and electric fields $\boldsymbol{F} = (\varepsilon_{ij}, -E_i)$ in the material with inhomogeneity satisfy the equations

$$\boldsymbol{F}(\mathbf{x}) = \boldsymbol{F}^0(\mathbf{x}) + \int_V \mathbf{P}(\mathbf{x}-\mathbf{x}')\mathbf{L}(\mathbf{x}')\,\mathrm{d}\mathbf{x}' + \omega^2\rho_1\int_V \mathbf{S}(\mathbf{x}-\mathbf{x}')\mathbf{J}\mathbf{f}(\mathbf{x}')\,\mathrm{d}\mathbf{x}',$$
$$\boldsymbol{F}^0(\mathbf{x}) = \begin{pmatrix} \varepsilon^0_{ij}(\mathbf{x}) \\ -E^0_i(\mathbf{x}) \end{pmatrix}, \quad \mathbf{P}(\mathbf{x}) = \begin{pmatrix} G_{i)(k,l)(j}(\mathbf{x}) & \gamma_{i),k(j}(\mathbf{x}) \\ \gamma_{k,il}(\mathbf{x}) & g_{,ik}(\mathbf{x}) \end{pmatrix} \tag{22}$$

when $\mathbf{x} \in V$ Eqs. (19) and (22) describe the electroelastic fields inside of the inhomogeneity from which the fields outside of it can be uniquely constructed.

3. Electroelastic fields in the transversely isotropic piezoelectric medium containing one continuous cylindrical fiber

We consider an inhomogeneity having the shape of an infinite circular cylinder (continuous fiber) with the axis parallel to the x_3-axis of the Cartesian coordinate system (Fig. 1).

Consider a plane wave propagating in the direction normal to x_3-axis. Since $\mathbf{L}(\mathbf{x})$ and $\rho(\mathbf{x})$ are functions of x_1, x_2 only, the fields $\mathbf{f}^0(\mathbf{x}), \mathbf{f}(\mathbf{x}), \boldsymbol{F}(\mathbf{x})$ are independent of x_3. Taking into account the relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k_3x_3'}\,\mathrm{d}x_3' = \delta(k_3) \tag{23}$$

Eqs. (19) and (22) are transformed into

$$\begin{aligned} \mathbf{f}(\mathbf{y}) &= \mathbf{f}^0(\mathbf{y}) + \int_S \mathbf{S}(\mathbf{y}-\mathbf{y}')\mathbf{L}^1\boldsymbol{F}(\mathbf{y}')\,\mathrm{d}\mathbf{y}' + \omega^2\rho_1\int_S \mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{J}\mathbf{f}(\mathbf{y}')\,\mathrm{d}\mathbf{y}' \\ \boldsymbol{F}(\mathbf{y}) &= \boldsymbol{F}^0(\mathbf{y}) + \int_S \mathbf{P}(\mathbf{y}-\mathbf{y}')\mathbf{L}^1\boldsymbol{F}(\mathbf{y}')\,\mathrm{d}\mathbf{y}' + \omega^2\rho_1\int_S \mathbf{S}(\mathbf{y}-\mathbf{y}')\mathbf{J}\mathbf{f}(\mathbf{y}')\,\mathrm{d}\mathbf{y}' \end{aligned} \tag{24}$$

where S is the cylindrical cross-section, $\mathbf{y} = (x_1, x_2)$ and

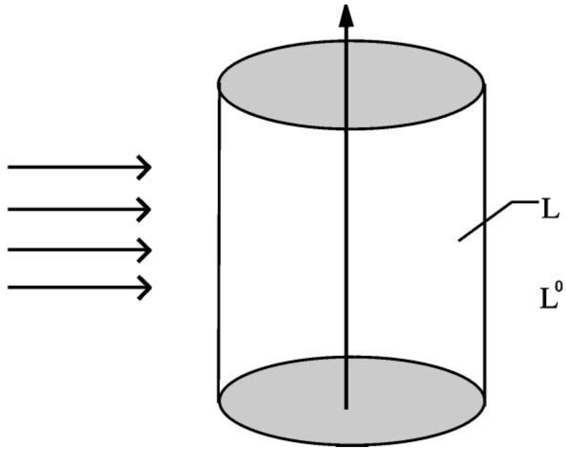

Fig. 1. Schematic: continuous fiber embedded in an infinite matrix subjected to an electroacoustic incident wave field propagating perpendicular to the fiber axis.

$$\mathbf{G}(\mathbf{y}-\mathbf{y}') = \frac{1}{(2\pi)^2}\int_0^\infty \bar{k}\,\mathrm{d}\bar{k}\int_0^{2\pi}\mathbf{G}(\bar{\mathbf{k}},\omega)\exp(-\mathrm{i}\bar{\mathbf{k}}\cdot(\mathbf{y}-\mathbf{y}'))\,\mathrm{d}\phi, \quad \bar{\mathbf{k}} = (k_1,k_2) \tag{25}$$

The expression for $\mathbf{G}(\bar{\mathbf{k}},\omega)$ is obtained from $\mathbf{G}(\mathbf{k},\omega)$ given by (13) by putting $k_3 = 0$. Let us assume that the matrix is transversely isotropic with the symmetry axis x_3. The material is characterized by five independent elastic moduli $\boldsymbol{C}^0 = \{C^0_{11}, C^0_{12}, C^0_{13}, C^0_{33}, C^0_{44}, C^0_{66} = (C^0_{11} - C^0_{12})/2\}$, three piezoelectric constants $\boldsymbol{e}^0 = \{e^0_{31}, e^0_{15}, e^0_{33}\}$ and two permeability coefficients $\boldsymbol{\eta}^0 = \{\eta^0_{11}, \eta^0_{33}\}$. The fiber material also possesses transversely isotropic symmetry with the same orientation as the matrix material (Fig. 1). We denote the tensors of elastic moduli, piezoelectric constants and permeability coefficients of the fibers by the same letters without the superscript "0". For the transversely isotropic matrix one obtains

$$\begin{aligned} \Lambda_{ik}(\mathbf{k}) &= \Lambda_1 n_i n_k + \Lambda_2(\theta_{ik} - n_i n_k) + \Lambda_3 m_i m_k \\ h_i(\mathbf{k}) &= h_i^{\mathrm{T}}(\mathbf{k}) = k^2 e^0_{15} m_i, \quad \lambda(\mathbf{k}) = k^2\eta^0_{11} \end{aligned} \tag{26}$$

where

$$\Lambda_1 = k^2 C^0_{11} - \rho_0\omega^2, \quad \Lambda_2 = k^2 C^0_{66} - \rho_0\omega^2, \quad \Lambda_3 = k^2 C^0_{44} - \rho_0\omega^2 \tag{27}$$

In these formulas m_i is the unit vector of x_3-axis and $\theta_{ij} = \delta_{ij} - m_i m_j$ is the "plane" Kronecker's delta. Here and in what follows the Fourier vector $\mathbf{k}$ always is $\mathbf{k} = (k_1, k_2)$ and the notation $n_i = k_i/|k|$ for the unit vector in $\mathbf{k}$-direction is used.

Expressions (26), (27) and (12) imply that

$$\begin{aligned} G_{ik}(\mathbf{k},\omega) &= \frac{1}{\Lambda_1} n_i n_k + \frac{1}{\Lambda_2}(\theta_{ik} - n_i n_k) + \frac{1}{\Lambda_3'} m_i m_k \\ \gamma_i(\mathbf{k}) &= \frac{e^0_{15}}{\eta^0_{11}\Lambda_3'} m_i, \quad g(\mathbf{k},\omega) = -\frac{1}{k^2\eta^0_{11}}\left[1 - \frac{k^2(e^0_{15})^2}{\eta^0_{11}\Lambda_3'}\right] \\ \Lambda_3' &= k^2 C'_{44} - \rho_0\omega^2, \quad C'_{44} = C^0_{44} + \frac{(e^0_{15})^2}{\eta^0_{11}} \end{aligned} \tag{28}$$

Introducing the quantities

$$\alpha^2 = \frac{\rho_0\omega^2}{C_{11}^0}, \quad \beta_1^2 = \frac{\rho_0\omega^2}{C_{66}^0}, \quad \beta_2^2 = \frac{\rho_0\omega^2}{C'_{44}} \tag{29}$$

the expressions (28) are recast as

$$\begin{aligned} G_{ik}(\mathbf{k},\omega) &= \frac{1}{\rho_0\omega^2}\left[\frac{\beta_1^2}{k^2-\beta_1^2}\theta_{ik} + k_i k_k\left(\frac{1}{k^2-\alpha^2} - \frac{1}{k^2-\beta_1^2}\right) + m_i m_k \frac{\beta_2^2}{k^2-\beta_2^2}\right] \\ g(\mathbf{k},\omega) &= -\frac{1}{\eta_{11}^0}\frac{1}{k^2} + \frac{1}{\rho_0\omega^2}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)^2 \frac{\beta_2^2}{k^2-\beta_2^2}, \quad \gamma_i(\mathbf{k},\omega) = \frac{1}{\rho_0\omega^2}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)\frac{\beta_2^2}{k^2-\beta_2^2} m_i \end{aligned} \tag{30}$$

To determine the **x**-representation of functions $G_{ik}(\mathbf{k},\omega)$, $\gamma_i(\mathbf{k},\omega)$ and $g(\mathbf{k},\omega)$, according to (14), we have to calculate an integral of the type

$$I = \frac{1}{(2\pi)^2}\int_0^\infty \frac{k\,\mathrm{d}k}{k^2-\beta^2}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{y}}\,\mathrm{d}\phi \tag{31}$$

To regularize this integral we introduce here an infinitesimal constant $\epsilon \to 0+$ with $\beta = (\omega/c) + \mathrm{i}\epsilon$. This step shifts the zeros of the denominator of the integrand into the complex plane to make this integral well defined. Appendix A shows that this regularization procedure corresponds to an infinitesimal damping and introduces *causality* (see Eq. (A.5)). Hence this regularization method has a strong physical motivation.

We have

$$\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{y}}\,\mathrm{d}\phi = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ky\cos\phi}\,\mathrm{d}\phi = 2\int_0^{\pi}\cos(ky\cos\phi)\,\mathrm{d}\phi = 2\pi J_0(ky) \tag{32}$$

where $J_0(z)$ is Bessel's function. In Appendix A it is outlined in detail (see Eqs. (A.5)–(A.20)) that

$$I = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^\infty \frac{J_0(ky)k\,\mathrm{d}k}{k^2-\beta^2} = \frac{\mathrm{i}}{4}H_0^{(1)}(\beta y) \tag{33}$$

Here $H_0^{(1)}(z)$ is Hankel's function of the first kind. Hence, the **x**-ω-representation of the Green's functions (30) has the form (Levin and Michelitsch, 1999)

$$\begin{aligned} G_{ik}(r,\omega) &= \frac{\mathrm{i}}{4\rho_0\omega^2}\left\{\theta_{ik}\beta_1^2 H_0^{(1)}(\beta_1 r) - \frac{\partial^2}{\partial y_i \partial y_k}\left[H_0^{(1)}(\alpha r) - H_0^{(1)}(\beta_1 r)\right] + m_i m_k \beta_2^2 H_0^{(1)}(\beta_2 r)\right\} \\ \gamma_i(r,\omega) &= \frac{\mathrm{i}}{4\rho_0\omega^2}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)\beta_2^2 H_0^{(1)}(\beta_2 r) m_i \\ g(r,\omega) &= \frac{1}{2\pi\eta_{11}^0}\ln r + \frac{\mathrm{i}}{4\rho_0\omega^2}\left(\frac{e_{15}^0}{\eta_{11}^0}\right)^2 \beta_2^2 H_0^{(1)}(\beta_2 r) \end{aligned} \tag{34}$$

where $r = |y|$.

A derivation of this Green's function of both the **x**–ω and the corresponding causal **x**–t representation is given in Appendix A. It follows from the structure of Eqs. (24) and (25) that electroelastic coupling does not influence the propagation of longitudinal and shear waves polarized in the x_1–x_2-plane. This is to be expected since this plane is the plane of isotropy so that the piezoelectric behavior does not manifest itself. The situation, however, is quite different when shear waves, polarized in x_3-direction ("axial" shear waves) propagate through the medium with inhomogeneity. The propagation of pure elastic waves in the medium with fiber inclusion was studied in a series of publications, (e.g. Achenbach, 1973; Every and Kim, 1995;

Tewary and Fortunko, 1992; Talbot and Willis, 1983). In the subsequent part we consider in detail *only* the propagation of *axial* shear waves. If we introduce the notation

$$u(\mathbf{y}) = u_3(\mathbf{y}), \quad \varepsilon_k(y) = \frac{\partial u_3(\mathbf{y})}{\partial y_k}, \quad C^0_{44} = \mu_0, \quad e^0_{15} = e_0, \quad \eta^0_{11} = \eta_0 \tag{35}$$

The elastic displacement $u(\mathbf{y})$ and electric potential $\varphi(\mathbf{y})$ satisfy the system of Eqs. (24) and (25) in which we put

$$\begin{aligned}
&\mathbf{f}(\mathbf{y}) = \begin{pmatrix} u(\mathbf{y}) \\ \varphi(\mathbf{y}) \end{pmatrix}, \quad \boldsymbol{F}(\mathbf{y}) = \begin{pmatrix} \varepsilon_k(\mathbf{y}) \\ -E_k(\mathbf{y}) \end{pmatrix}, \quad \mathbf{L}^1 = \begin{pmatrix} \mu_1 & e_1 \\ e_1 & -\eta_1 \end{pmatrix} \\
&\mathbf{G}(R) = \frac{1}{2\pi\eta_0}\mathbf{T}_2 \ln R + \frac{1}{\overline{\mu}_0}\mathbf{T}_3 G(R), \quad G(R) = \frac{\mathrm{i}}{4} H^{(1)}_0(k_0 R), \quad \mathbf{J} = \mathbf{T}_1 \\
&\mathbf{T}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{T}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{T}_3 = \begin{pmatrix} 1 & e_0/\eta_0 \\ e_0/\eta_0 & (e_0/\eta_0)^2 \end{pmatrix} \\
&k_0 = \beta_2, \quad R = |\mathbf{y} - \mathbf{y}'|, \quad \mu_1 = C_{44} - C^0_{44}, \quad e_1 = e_{15} - e^0_{15} \\
&\eta_1 = \eta_{11} - \eta^0_{11}, \quad \overline{\mu}_0 = \mu_0 + \frac{e_0^2}{\eta_0}
\end{aligned} \tag{36}$$

The system of integral Eqs. (24) and (25) is difficult to solve exactly, so instead we restrict ourselves to the *long-wave approximation*. If the wavelengths of the incident fields are much larger than the fiber diameter a we can suppose that the change of the fields $\mathbf{f}(\mathbf{y})$ and $\boldsymbol{F}(\mathbf{y})$ inside of region S can be neglected. It gives

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}^0(\mathbf{y}) + \nabla\mathbf{g}(\mathbf{y})\mathbf{L}^1\boldsymbol{F}(\mathbf{y}) + \rho_1\omega^2\mathbf{g}(\mathbf{y})\mathbf{T}_1\mathbf{f}(\mathbf{y}) \tag{37}$$

$$\boldsymbol{F}(\mathbf{y}) = \boldsymbol{F}^0(\mathbf{y}) + \mathbf{P}(\mathbf{y})\mathbf{L}^1\boldsymbol{F}(\mathbf{y}) + \rho_1\omega^2\nabla\mathbf{g}(\mathbf{y})\mathbf{T}_1\mathbf{f}(\mathbf{y}) \tag{38}$$

where it is denoted

$$\mathbf{g}(\mathbf{y}) = \int_S \mathbf{G}(\mathbf{y} - \mathbf{y}')\,\mathrm{d}\mathbf{y}', \quad \mathbf{P}(\mathbf{y}) = \nabla \otimes \nabla\mathbf{G}(\mathbf{y}) \tag{39}$$

$\mathbf{g}(\mathbf{y})$ is the integral of Green's function over the inhomogeneity S and can be interpreted as the *Green's function* corresponding to a spatial source distribution which is represented by the inhomogeneity S, i.e. the fiber cross-section with radius a.

During the integration of function $\mathbf{G}$ over the region S let us take into account that

$$\int_S \ln|\mathbf{y} - \mathbf{y}'|\,\mathrm{d}\mathbf{y}' = \frac{\pi}{2}\left[r^2 - a^2(1 - 2\ln a)\right], \quad r = |\mathbf{y}| \tag{40}$$

where a is the fiber radius. This integral is derived in Appendix C. To calculate the integral of $G(y)$ over the inclusion it is convenient to use the Fourier transform

$$G(y, a) = \frac{\mathrm{i}}{4}\int_S H^{(1)}_0(k_0|\mathbf{y} - \mathbf{y}'|)\,\mathrm{d}\mathbf{y}' = \frac{1}{(2\pi)^2}\int \mathrm{d}\mathbf{k}\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{y}} G(\mathbf{k}) \int_S \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}'}\,\mathrm{d}\mathbf{y}' \tag{41}$$

An extensive derivation of this integral and a discussion of its properties and physical meaning is given in Appendix B. Because of

$$\int_S \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}'}\,\mathrm{d}\mathbf{y}' = \frac{2\pi a}{k} J_1(ka), \quad \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}'}\,\mathrm{d}\phi = 2\pi J_0(kr) \tag{42}$$

we obtain

$$G(y,a) = a\int_0^\infty \frac{J_0(kr)J_1(ka)\,\mathrm{d}k}{k^2 - k_0^2} \tag{43}$$

This expression is evaluated in detail in Appendix B. Again we employ there the regularization method by introducing an infinitesimal damping constant $\epsilon \to 0+$ according to ($k_0 = \mathrm{Re}k_0 + \mathrm{i}\epsilon$).

Via the representation

$$\int_0^\infty \frac{J_0(kr)J_1(ka)\,\mathrm{d}k}{k^2 - k_0^2} = \frac{1}{k_0^2}\left\{ -\frac{\partial}{\partial a}\int_0^\infty \frac{J_0(kr)J_0(ka)k\,\mathrm{d}k}{k^2 - k_0^2} - \int_0^\infty J_0(kr)J_1(ka)\,\mathrm{d}k \right\} \tag{44}$$

the integral in the right-hand side of (43) is transformed into two simplest ones which yield

$$\int_0^\infty J_0(kr)J_1(ka)\,\mathrm{d}k = \frac{1}{a}, \quad \int_0^\infty \frac{J_0(kr)J_0(ka)k\,\mathrm{d}k}{k^2 - k_0^2} = \lim_{\epsilon\to 0}\lim_{R\to\infty}\int_0^R \frac{J_0(kr)J_0(ka)k\,\mathrm{d}k}{k^2 + (\mathrm{i}k_0)^2} = I_0(-\mathrm{i}k_0 r)K_0(-\mathrm{i}ka) \tag{45}$$

where $I_n(z)$ and $K_n(z)$ are the modified Bessel's functions of the first and second kind.

Finally, we have for (43) ($r < a$)

$$G(y,a) = \frac{1}{k_0^2}\left[\frac{\mathrm{i}\pi}{2}J_0(k_0 r)k_0 a H_1^{(1)}(k_0 a) - 1\right] \tag{46}$$

A detailed derivation of (46) is given in Appendix B. According to (36) we can write

$$\begin{aligned} \partial_j \mathbf{g}(\mathbf{y}) &= \left[\frac{1}{2\eta_0}\mathbf{T}_2 - \frac{\pi \mathrm{i}}{2\bar{\mu}_0}\mathbf{T}_3 J_1(k_0 r)\frac{a}{r}H_1^{(1)}(k_0 a)\right]y_j \\ \mathbf{P}(\mathbf{y}) &= \frac{1}{2\eta_0}\mathbf{T}_2\theta_{ij} - \frac{1}{\bar{\mu}_0}\mathbf{T}_3\left[\frac{J_1(k_0 r)}{k_0 r}\theta_{ij} - J_2(k_0 r)n_i n_j\right]\frac{\mathrm{i}\pi}{2}k_0 a H_1^{(1)}(k_0 a), \quad n_i = y_i/|\mathbf{y}| \end{aligned} \tag{47}$$

Let the incident fields $u^0(\mathbf{y})$ and $\varphi^0(\mathbf{y})$ be plane axial shear waves

$$u^0(\mathbf{y}) = U^0 \mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{y}}, \quad \varphi^0(\mathbf{y}) = \Phi^0 \mathrm{e}^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{y}} \tag{48}$$

Because of the equation

$$e_0 \Delta u(\mathbf{y}) - \eta_0 \Delta\varphi(\mathbf{y}) = 0 \tag{49}$$

the amplitude of the electric potential Φ^0 is expressed via the amplitude of the elastic displacement U^0

$$\Phi^0 = \frac{e_0}{\eta_0}U^0 \tag{50}$$

If $\mathbf{y} \in S$ we have in the long-wave limit $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}} \approx 1$ and the gradients of these fields $\varepsilon_k^0(\mathbf{y})$ and $E_k^0(\mathbf{y})$ have the order ω. We can now expand the Bessel's functions in (46) and (47) into asymptotic series when their arguments are small. In what follows we will take into account only the main terms of this expansion: the terms that are constant (independent on ω) in the real parts of all expressions and the terms having the order ω^2 in the imaginary parts. [1] With the help of asymptotic formulas

$$J_n(z) \sim \frac{1}{n!}\left(\frac{z}{2}\right)^n, \quad \frac{\mathrm{i}\pi}{2}z^n H_n^{(1)}(z) \sim 2^{n-1}\left[(n-1)! + \frac{\mathrm{i}\pi}{n!}\left(\frac{z}{2}\right)^{2n}\right] \tag{51}$$

[1] As it is well known such approximation will allow to describe the attenuation of the electroacoustic waves but not the dispersion.

we can write the system (37) in the form of two equations

$$u = U^0 + \frac{\rho_1}{\rho_0} \cdot \frac{i\pi}{4} (k_0 a)^2 u$$
$$\varphi = \Phi^0 + \frac{\rho_1}{\rho_0} \cdot \frac{i\pi}{4} (k_0 a)^2 \varphi \tag{52}$$

the solution of which with prescribed accuracy is

$$u = \left[1 - \frac{\rho_1}{\rho_0} \cdot \frac{i\pi}{4} (k_0 a)^2\right] U^0$$
$$\varphi = \frac{e_0}{\eta_0}\left[1 - \frac{\rho_1}{\rho_0} \cdot \frac{i\pi}{4} (k_0 a)^2\right] U^0 \tag{53}$$

The second system of equations takes the form

$$\boldsymbol{F} = \boldsymbol{F}^0 + \left[\mathbf{P}^R + \frac{i\pi}{4} (k_0 a)^2 \mathbf{P}^I\right] \mathbf{L}^1 \boldsymbol{F} \tag{54}$$

where it is denoted

$$\mathbf{P}^R = \frac{1}{2}\left(\frac{1}{\eta_0}\mathbf{T}_2 - \frac{1}{\bar{\mu}_0}\mathbf{T}_3\right) \otimes \boldsymbol{\theta}, \quad \mathbf{P}^I = -\frac{1}{2\bar{\mu}_0}\mathbf{T}_3 \otimes \boldsymbol{\theta} \tag{55}$$

where $\boldsymbol{\theta} = (\theta_{ij}) = (\delta_{ij} - m_i m_j)$. The solution of this system with the same accuracy is

$$\boldsymbol{F} = \left[\mathbf{A} - \frac{i\pi}{4} (k_0 a)^2 \mathbf{B}\right] \boldsymbol{F}^0,$$
$$\mathbf{A} = \left(\mathbf{I} - \mathbf{P}^R \mathbf{L}^1\right)^{-1}, \quad \mathbf{B} = -\mathbf{A}\mathbf{P}^I \mathbf{L}^1 \mathbf{A} \tag{56}$$

The calculation of matrices **A** and **B** with the help of (55) gives

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{57}$$

where

$$a_{11} = \frac{2}{\Delta}[e_0(2e_0 + e_1) + \mu_0(2\eta_0 + \eta_1)]$$
$$a_{12} = \frac{2}{\Delta}(e_0\eta_1 - \eta_0 e_1)$$
$$a_{21} = \frac{2}{\Delta}(\mu_0 e_1 - e_0\mu_1)$$
$$a_{22} = \frac{2}{\Delta}[e_0(2e_0 + e_1) + \eta_0(2\mu_0 + \mu_1)] \tag{58}$$

and

$$b_{11} = \frac{2\bar{\mu}_0}{\Delta^2}(2\eta_0 + \eta_1)[e_1(2e_0 + e_1) + \mu_1(2\eta_0 + \eta_1)]$$
$$b_{12} = \frac{4\bar{\mu}_0}{\Delta^2}(2\eta_0 + \eta_1)(e_1\eta_0 - \eta_1 e_0)$$
$$b_{21} = \frac{2\bar{\mu}_0}{\Delta^2}(2e_0 + e_1)[e_1(2e_0 + e_1) + \mu_1(2\eta_0 + \eta_1)]$$
$$b_{22} = \frac{4\bar{\mu}_0}{\Delta^2}(2e_0 + e_1)(e_1\eta_0 - e_0\eta_1) \tag{59}$$

In these expressions we have put

$$\Delta = (2\mu_0 + \mu_1)(2\eta_0 + \eta_1) + (2e_0 + e_1)^2 \tag{60}$$

Let us note that for the "static" case ($k_0 = 0$) the expression (56) (with the components a_{ij}) is in agreement with the expressions obtained by Levin et al. (2000). We denote now the obtained approximated solution (53) and (56) as

$$\mathbf{f}^{(0)} = \lambda \mathbf{f}^0, \quad \boldsymbol{F}^{(0)} = \boldsymbol{\Lambda} \boldsymbol{F}^0, \quad \lambda = 1 + \frac{\rho_1}{\rho_0}\frac{\mathrm{i}\pi}{4}(k_0 a)^2, \quad \boldsymbol{\Lambda} = \mathbf{A} - \frac{\mathrm{i}\pi}{4}(k_0 a)^2 \mathbf{B} \tag{61}$$

and substitute these expressions into the initial integral equations. In symbolic form we can write

$$\begin{aligned} &\lambda \boldsymbol{f}^0 - \nabla \mathbf{g} \mathbf{L}^1 \boldsymbol{\Lambda} \boldsymbol{F}^0 - \rho_1 \omega^2 \mathbf{g} \mathbf{T}_1 \lambda \mathbf{f}^0 = \mathbf{f}^0 + \Delta_1 \\ &\boldsymbol{\Lambda} \boldsymbol{F}^0 - \mathbf{P} \mathbf{L}^1 \boldsymbol{\Lambda} \boldsymbol{F}^0 - \rho_1 \omega^2 \nabla \mathbf{g} \mathbf{T}_1 \lambda \mathbf{f}^0 = \boldsymbol{F}^0 + \Delta_2 \end{aligned} \tag{62}$$

where Δ_1 and Δ_2 are the discrepancies due to proximity of expressions (61). To compensate these discrepancies it is necessary to add to $\mathbf{f}^{(0)}$ and $\boldsymbol{F}^{(0)}$ the items $\mathbf{f}^{(1)}$ and $\boldsymbol{F}^{(1)}$ in such a way that the following equations are satisfied

$$\begin{aligned} &\mathbf{f}^{(1)} - \nabla \mathbf{g} \mathbf{L}^1 \boldsymbol{F}^{(1)} - \rho_1 \omega^2 \mathbf{g} \mathbf{T}_1 \mathbf{f}^{(1)} = \Delta_1 \\ &\boldsymbol{F}^{(1)} - \mathbf{P} \mathbf{L}^1 \boldsymbol{F}^{(1)} - \rho_1 \omega^2 \nabla \mathbf{g} \mathbf{T}_1 \mathbf{f}^{(1)} = \Delta_2 \end{aligned} \tag{63}$$

If we can neglect the quantities $\mathbf{f}^{(1)}$ and $\boldsymbol{F}^{(1)}$ in comparison with (61) then functions $\mathbf{f}^{(0)}$ and $\boldsymbol{F}^{(0)}$ are really the main terms of expansion of the solution of the initial equations in the series with respect to parameter $k_0 a$. Otherwise we have to add to (61) the main terms of analogous expansion of functions $\mathbf{f}^{(1)}$ and $\boldsymbol{F}^{(1)}$.

In our case Δ_1 and Δ_2 have the order $\mathrm{O}((k_0 a)^2)$. Therefore functions $\mathbf{f}^{(1)}$ and $\boldsymbol{F}^{(1)}$ have the order at least $\mathrm{O}((k_0 a)^3)$ and can be neglected in comparison with $\mathbf{f}^{(0)}$ and $\boldsymbol{F}^{(0)}$.

4. Analogue of optical theorem and total scattering cross-section

As it follows from Eq. (24) the electroelastic fields in the matrix can be represented in the form

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}^0(\mathbf{y}) + \mathbf{f}^{\mathrm{s}}(\mathbf{y}) \tag{64}$$

where $\mathbf{f}^{\mathrm{s}}(\mathbf{y}) = (u^{\mathrm{s}}(\mathbf{y}), \varphi^{\mathrm{s}}(\mathbf{y}))$ are the scattering fields that are determined by the expression

$$\mathbf{f}^{\mathrm{s}}(\mathbf{y}) = \int_S [\nabla \mathbf{G}(\mathbf{y} - \mathbf{y}') \mathbf{L}^1 \boldsymbol{F}(\mathbf{y}') + \rho_1 \omega^2 \mathbf{G}(\mathbf{y} - \mathbf{y}') \mathbf{T}_1 \mathbf{f}(\mathbf{y}')] \, \mathrm{d}\mathbf{y}' \tag{65}$$

or in the more details

$$\begin{aligned} u^{\mathrm{s}}(\mathbf{y}) &= \int_S [\Psi_k(R) \varepsilon_k(\mathbf{y}') - \psi_k(R) E_k(\mathbf{y}') + \rho_1 \omega^2 G(R) u(\mathbf{y}')] \, \mathrm{d}\mathbf{y}' \\ \varphi^{\mathrm{s}}(\mathbf{y}) &= \int_S \left[\Phi_k(R) \varepsilon_k(\mathbf{y}') - \phi_k(R) E_k(\mathbf{y}') + \rho_1 \omega^2 \frac{e_0}{\eta_0} G(R) u(\mathbf{y}')\right] \mathrm{d}\mathbf{y}' \end{aligned} \tag{66}$$

Here it is denoted $R = |\mathbf{y} - \mathbf{y}'|$ and

$$
\begin{aligned}
\Psi_k(R) &= \left(\mu_1 + \frac{e_0}{\eta_0} e_1\right) \nabla_k G(R), \\
\psi_k(R) &= \left(e_1 - \frac{e_0}{\eta_0} e_1\right) \nabla_k G(R), \\
\Phi_k(R) &= \frac{e_0}{\eta_0} \mu_1 \nabla_k G(R) + e_1 \nabla_k g(R), \\
\phi_k(R) &= \frac{e_0}{\eta_0} e_1 \nabla_k G(R) - \eta_1 \nabla_k g(R)
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
G(R) &= \frac{\mathrm{i}}{4\overline{\mu}_0} H_0^{(1)}(k_0 R), \quad \overline{\mu}_0 = \mu_0 + \frac{e_0^2}{\eta_0} \\
g(R) &= \frac{1}{2\pi\eta_0} \ln R + \left(\frac{e_0}{\eta_0}\right)^2 \frac{\mathrm{i}}{4\overline{\mu}_0} H_0^{(1)}(k_0 R)
\end{aligned}
\tag{68}
$$

Expressions (66) allow us to find the far-field asymptotics of the electroelastic fields. Taking into account the asymptotic formulas at $R \to \infty$ [2]

$$
|\mathbf{y} - \mathbf{y}'|^{-1} \sim y^{-1}, \quad |\mathbf{y} - \mathbf{y}'| \sim y - (\mathbf{n} \cdot \mathbf{y}'), \quad n_i = \frac{y_i}{y}, \quad y = |\mathbf{y}| \tag{69}
$$

$$
\frac{\partial}{\partial y_{k_1}} \cdots \frac{\partial}{\partial y_{k_m}} H_0^{(1)}(qR) \sim (\mathrm{i}q)^m n_{k_1} \cdots n_{k_m} \sqrt{\frac{2}{\pi q y}} \mathrm{e}^{\mathrm{i}\left(qy - \frac{\pi}{4}\right)} \mathrm{e}^{-\mathrm{i}q(\mathbf{n}\cdot\mathbf{y}')} \tag{70}
$$

we can write

$$
u^{\mathrm{s}}(\mathbf{y}) = C(\mathbf{n}) \frac{\mathrm{e}^{\mathrm{i}k_0 y}}{\sqrt{y}}, \quad \varphi^{\mathrm{s}}(\mathbf{y}) = \frac{e_0}{\eta_0} C(\mathbf{n}) \frac{\mathrm{e}^{\mathrm{i}k_0 y}}{\sqrt{y}} \tag{71}
$$

Here $C(\mathbf{n})$ is the amplitude of cylindrical waves that can be represented in the form

$$
\begin{aligned}
C(\mathbf{n}) = \frac{\mathrm{i}}{2\rho_0\omega^2} \sqrt{\frac{k_0^3}{2\pi}} \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} \Bigg[& \mathrm{i}k_0 n_k \left(\mu_1 + \frac{e_0}{\eta_0} e_1\right) \int_S \varepsilon_k(\mathbf{y}') \mathrm{e}^{-\mathrm{i}k_0(\mathbf{n}\cdot\mathbf{y}')} \, \mathrm{d}\mathbf{y}' \\
& - \mathrm{i}k_0 n_k \left(e_1 - \frac{e_0}{\eta_0} \eta_1\right) \int_S E_k(\mathbf{y}') \mathrm{e}^{-\mathrm{i}k_0(\mathbf{n}\cdot\mathbf{y}')} \, \mathrm{d}\mathbf{y}' + \rho_1 \omega^2 \int_S u(\mathbf{y}') \mathrm{e}^{-\mathrm{i}k_0(\mathbf{n}\cdot\mathbf{y}')} \mathrm{d}\mathbf{y}' \Bigg]
\end{aligned}
\tag{72}
$$

Let us find now the asymptotic expressions for the gradients of the scattered electroelastic fields

$$
\varepsilon_k^{\mathrm{s}} = \mathrm{i}k_0 n_k C(\mathbf{n}) \frac{\mathrm{e}^{\mathrm{i}k_0 y}}{\sqrt{y}}, \quad E_k^{\mathrm{s}} = -\mathrm{i}k_0 n_k \frac{e_0}{\eta_0} C(\mathbf{n}) \frac{\mathrm{e}^{\mathrm{i}k_0 y}}{\sqrt{y}} \tag{73}
$$

and corresponding fields of the stress σ_k^{s} and electric displacement D_k^{s}

$$
\begin{aligned}
\sigma_k^{\mathrm{s}} &= \mu_0 \varepsilon_k^{\mathrm{s}} - e_0 E_k^4 = \mathrm{i}k_0 \overline{\mu}_0 n_k C(\mathbf{n}) \frac{\mathrm{e}^{\mathrm{i}k_0 y}}{\sqrt{y}} \\
D_k^{\mathrm{s}} &= e_0 \varepsilon_k^{\mathrm{s}} + \eta_0 E_k^{\mathrm{s}} \equiv 0
\end{aligned}
\tag{74}
$$

[2] A general derivation of far-field asymptotics is given in Appendix D.

Let us suppose that the incident fields have the form

$$u^0(\mathbf{y}) = U^0 e^{ik_0\mathbf{n}_0\cdot\mathbf{y}}, \quad \varphi^0(\mathbf{y}) = \Phi^0 e^{ik_0\mathbf{n}_0\cdot\mathbf{y}} \tag{75}$$

Because of equation

$$e_0\Delta u^0(\mathbf{y}) - \eta_0\Delta\varphi^0(\mathbf{y}) = 0$$

we have

$$\Phi^0 = \frac{e_0}{\eta_0}U^0 \tag{76}$$

and

$$\sigma_k^0 = ik_0\overline{\mu}_0 n_k^0 U^0 e^{ik_0\mathbf{n}_0\cdot\mathbf{y}}, \quad D_k^0 = 0 \tag{77}$$

We define the intensity vector I_k associated with a stress field σ_k and the velocity $\dot{u}$ of the considered axial shear wave by the relation

$$I_k = \sigma_k\dot{u} \tag{78}$$

Similarly, we denote by I_k^s the intensity vector associated with the scattered fields, and by I_k^0 the intensity vector associated with the incident fields. The term "intensity" refers to the rate of energy transfer per unit area in the direction normal to the one of propagation, that is

$$I = I_k n_k \tag{79}$$

where n_k is the unit vector in the direction of propagation. The power flux (the rate of energy transfer across the surface S with unit normal n_i) is

$$Q = \int_S I_k n_k \, dS = \int_S \sigma_k \dot{u} n_k \, dS \tag{80}$$

For a given angular frequency corresponding to period T the total scattering cross-section $Q(\omega)$ is the ratio of the average power flux over all directions to the average intensity of the incident fields

$$Q(\omega) = \frac{\langle Q^s\rangle_t}{\langle I^0\rangle_t} \tag{81}$$

where $\langle\cdot\rangle_t$ denotes the time averaging over the period T.

Having found the far-field asymptotics of the scattered electroelastic fields we can now compute the total scattering cross-section according to relation (81). The power flux is a real number defined by

$$\langle Q\rangle_t = \frac{1}{4}\int_S \left\langle (\sigma_k + \sigma_k^*)(\dot{u} + \dot{u}^*)\right\rangle_t n_k \, dS \tag{82}$$

where $*$ denotes the complex conjugate. Since we assume the vibrations are harmonic we can write

$$\langle Q\rangle_t = \frac{i\omega}{4}\int_S \left\langle \sigma_k u e^{2i\omega t} - \sigma_k^* u^* e^{-2i\omega t} + \sigma_k^* u - \sigma_k u^*\right\rangle_t n_k \, dS \tag{83}$$

Computing the time average gives

$$\langle Q\rangle_t = \frac{1}{2}\omega \mathrm{Im}\int_S \sigma_k u^* n_k \, dS \tag{84}$$

Taking into account that

$$\sigma_k = \sigma_k^0 + \sigma_k^s, \quad u = u^0 + u^s \tag{85}$$

let us represent this value as the sum of three items connecting with incident fields $\langle Q^0\rangle_t$, scattered fields $\langle Q^s\rangle_t$ and interference of the exciting and scattering fields $\langle Q^{int}\rangle_t$:

$$\langle Q\rangle_t = \langle Q^0\rangle_t + \langle Q^s\rangle_t + \langle Q^{int}\rangle_t \quad \langle Q^0\rangle_t = \frac{1}{2}\omega \mathrm{Im}\int_S \sigma_k^0 u^{0*} n_k \,\mathrm{d}S,$$
$$\langle Q^s\rangle_t = \frac{1}{2}\omega \mathrm{Im}\int_S \sigma_k^s u^{s*} n_k \,\mathrm{d}S \quad \langle Q^{int}\rangle_t = \frac{1}{2}\omega \mathrm{Im}\int_S \left(\sigma_k^0 u^{s*} + \sigma_k^s u^{0*}\right) n_k \,\mathrm{d}S \tag{86}$$

In view of the energy conservation law we have

$$\langle Q^s\rangle_t = -\langle Q^{int}\rangle_t = -\frac{1}{2}\omega \mathrm{Im}\int_S (\sigma_k^0 u^{s*} + \sigma_k^s u^{0*}) n_k \,\mathrm{d}S \tag{87}$$

Because of $\langle I^0\rangle_t = \frac{1}{2}\omega \mathrm{Im}[\sigma_k^0 u^{0*} n_k^0] = \frac{1}{2}\omega k_0 \overline{\mu}_0$, the total scattering cross-section is determined by the expression

$$Q(\omega) = \frac{\langle Q^s\rangle}{\langle I^0\rangle_t} = -\frac{\mathrm{Im} J(\omega)}{\overline{\mu}_0 k_0} \tag{88}$$

where it is denoted

$$J(\omega) = \int_S (\sigma_k^0 u^{s*} + \sigma_k^s u^{0*}) n_k \,\mathrm{d}S \tag{89}$$

and amplitude U^0 is taken equal to unity.

In the case considered, S is the cylindrical surface of a cylinder with large radius r and unit height coaxial with the fiber. Hence, integral (89) can be rewritten as

$$J(\omega) = r\int_0^{2\pi} (\sigma_k^0 u^{s*} + \sigma_k^s u^{0*}) n_k \,\mathrm{d}\phi, \quad r = |\mathbf{y}| \tag{90}$$

Using expressions (74)–(77) we obtain

$$(\sigma_k^0 u^{s*} + \sigma_k^s u^{0*}) n_k = \frac{\mathrm{i}k_0\overline{\mu}_0}{\sqrt{r}}\left[C^*(\mathbf{n})\mathrm{e}^{-\mathrm{i}k_0 r}\mathrm{e}^{\mathrm{i}k_0\mathbf{n_0}\cdot\mathbf{r}}(\mathbf{n_0}\cdot\mathbf{n}) + C(\mathbf{n})\mathrm{e}^{\mathrm{i}k_0 r}\mathrm{e}^{-\mathrm{i}k_0\mathbf{n_0}\cdot\mathbf{r}}\right] \tag{91}$$

Let us suppose that the incident field is a plane wave propagating in the direction opposite to the positive direction of x_1-axis. Then

$$\mathbf{n}_0 = (-1, 0), \quad \mathbf{n} = (\cos\phi, \sin\phi), \quad \mathbf{n_0}\cdot\mathbf{n} = -\cos\phi \tag{92}$$

and integral $J(\omega)$ takes the form

$$J(\omega) = \mathrm{i}k_0\overline{\mu}_0\sqrt{r}\left[\mathrm{e}^{\mathrm{i}k_0 r}\int_{-\phi_0}^{2\pi-\phi_0} C(\mathbf{n})\mathrm{e}^{\mathrm{i}k_0 r\cos\phi}\,\mathrm{d}\phi - \mathrm{e}^{-\mathrm{i}k_0 r}\int_{-\phi_0}^{2\pi-\phi_0} C^*(\mathbf{n})\mathrm{e}^{-\mathrm{i}k_0 r\cos\phi}\cos\phi\,\mathrm{d}\phi\right] \tag{93}$$

This integral is performed over a whole period. Angle ϕ_0 with $0 < \phi_0 < \pi$ is only introduced here to make this expression useful for the application of the method of stationary points. [3] The derivation of the main term of the asymptotics for $r \to \infty$ of such an integral is shown in Appendix D. To that end we consider the asymptotic representation of the integral

$$F(\lambda) = \int_a^b f(x)\exp[\mathrm{i}\lambda S(x)]\,\mathrm{d}x, \quad \lambda \to \infty \tag{94}$$

[3] Due to the 2π-periodicity of the integrands the value of this integral is independent of ϕ_0.

in the case when function $S(x)$ has m simple stationary points x_v ($v = 1, \dots, m$) with $a < x_v < b$ *inside* of interval $[a, b]$. [4] Then the leading term of the asymptotic expression (94) has the form

$$F(\lambda; \{x_v\}) = \sum_{v=1}^{m} \sqrt{\frac{2\pi}{\lambda |S''(x_v)|}} \left[f(x_v) + \mathrm{O}\left(\frac{1}{\lambda}\right) \right] \exp\left[\mathrm{i}\lambda S(x_v) + \frac{\mathrm{i}\pi}{4} \mathrm{sign}[S''(x_v)] \right] \tag{95}$$

In Appendix D is given a detailed derivation of this expression where the method of stationary points is applied.

In order to evaluate the farfield asymptotics of (93) we then have for $r \to \infty$

$$\begin{aligned} \int_0^{2\pi} C(\mathbf{n}) \mathrm{e}^{\mathrm{i}k_0 r \cos\phi} \mathrm{d}\phi &\sim \sqrt{\frac{2\pi}{k_0 r}} \left(C(\mathbf{n}_0) \mathrm{e}^{-\mathrm{i}k_0 r + \frac{\mathrm{i}\pi}{4}} + C(-\mathbf{n}_0) \mathrm{e}^{\mathrm{i}k_0 r - \frac{\mathrm{i}\pi}{4}} \right) \\ \int_0^{2\pi} C^*(\mathbf{n}) \cos\phi \, \mathrm{e}^{\mathrm{i}k_0 r \cos\phi} \mathrm{d}\phi &\sim \sqrt{\frac{2\pi}{k_0 r}} \left(-C^*(\mathbf{n}_0) \mathrm{e}^{\mathrm{i}k_0 r - \frac{\mathrm{i}\pi}{4}} + C^*(-\mathbf{n}_0) \mathrm{e}^{-\mathrm{i}k_0 r + \frac{\mathrm{i}\pi}{4}} \right) \end{aligned} \tag{96}$$

Hence,

$$\begin{aligned} J(\omega) &= \mathrm{i}\overline{\mu}_0 \sqrt{2\pi k_0} \left[C(\mathbf{n}_0) \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} + C^*(\mathbf{n}_0) \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} + C(-\mathbf{n}_0) \mathrm{e}^{2\mathrm{i}k_0 r - \frac{\mathrm{i}\pi}{4}} - C^*(-\mathbf{n}_0) \mathrm{e}^{-2\mathrm{i}k_0 r + \frac{\mathrm{i}\pi}{4}} \right] \\ &= 2\mathrm{i}\overline{\mu}_0 \sqrt{2\pi k_0} \mathrm{Re}\left[C(\mathbf{n}_0) \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \right] - 2\overline{\mu}_0 \sqrt{2\pi k_0} \mathrm{Im}\left[C(-\mathbf{n}_0) \mathrm{e}^{2\mathrm{i}k_0 r - \frac{\mathrm{i}\pi}{4}} \right] \end{aligned} \tag{97}$$

and according to (88) we obtain finally

$$Q(\omega) = -2\sqrt{\frac{2\pi}{k_0}} \mathrm{Re}\left[C(\mathbf{n}_0) \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \right] \tag{98}$$

The last equation is the analogue of the optical theorem in the theory of electromagnetic waves (see, for example Bohren and Huffmen (1983)).

In the long-wave limit we can put in the integrals of (72) $\exp(-\mathrm{i}k_0 \mathbf{n}_0 \cdot \mathbf{y}') \approx 1$. Taking into account that the fields ε_k, E_k and u are constant inside of region S we have

$$C(\mathbf{n}_0) \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} = \mathrm{i}\left\{ \frac{\pi a^2}{2\rho_0 \omega^2} \sqrt{\frac{k_0^3}{2\pi}} \left[\mathrm{i}k_0 n_k^0 \left(\mu_1 + \frac{e_0}{\eta_0} e_1 \right) \varepsilon_k - k_0 n_k^0 \left(e_1 - \frac{e_0}{\eta_0} \eta_1 \right) E_k + \rho_1 \omega^2 u \right] \right\} \tag{99}$$

It follows from here and (98) that

$$Q(\omega) = \frac{\pi a^2}{\rho_0 \omega^2} k_0 \mathrm{Im}\left[\mathrm{i}k_0 n_k^0 \left(\mu_1 + \frac{e_0}{\eta_0} e_1 \right) \varepsilon_k - \mathrm{i}k_0 n_k^0 \left(e_1 - \frac{e_0}{\eta_0} \eta_1 \right) E_k + \rho_1 \omega^2 u \right] \tag{100}$$

The expressions for the fields ε_k, E_k and u inside of region S in the long-wave approximation were obtained in the Section 3 in (53) and (56)–(59). According to these formulae

$$\begin{aligned} \mathrm{Im}\,\varepsilon_k &= -\frac{\pi}{4} (k_0 a)^2 \left(b_{11} \varepsilon_k^0 - b_{12} E_k^0 \right) \\ \mathrm{Im}\,E_k &= \frac{\pi}{4} (k_0 a)^2 \left(b_{21} \varepsilon_k^0 - b_{22} E_k^0 \right) \\ \mathrm{Im}\,u &= \frac{\pi}{4} (k_0 a)^2 \frac{\rho_1}{\rho_0} \end{aligned} \tag{101}$$

[4] It is assumed that the integration limits a, b are no stationary points.

where b_{ij} are determined in (59). After substitution of these expressions into the right-hand side of (100) and taking into account that in the long-wave limit

$$\varepsilon_k^0 = \mathrm{i}k_0 n_k^0, \quad E_k^0 = -\frac{e_0}{\eta_0}\mathrm{i}k_0 n_k^0 \tag{102}$$

we obtain

$$Q(\omega) = \frac{\pi^2 (k_0 a)^3}{8} a \left\{ \frac{1}{\bar{\mu}_0} \left[\mu_{\mathrm{B}} + 2\frac{e_0}{\eta_0} e_{\mathrm{B}} - \left(\frac{e_0}{\eta_0}\right)^2 \eta_{\mathrm{B}} \right] + 2\left(\frac{\rho_1}{\rho_0}\right)^2 \right\} \tag{103}$$

where it is denoted

$$\mu_{\mathrm{B}} = \mu_1 b_{11} + e_1 b_{21}, \quad e_{\mathrm{B}} = \mu_1 b_{12} + e_1 b_{22}, \quad \eta_{\mathrm{B}} = \eta_1 b_{22} - e_1 b_{12} \tag{104}$$

It can be shown that expression (103) can be rewritten in the form

$$Q(\omega) = \frac{\pi^2 (k_0 a)^3}{4} a \left\{ \frac{1}{2\bar{\mu}_0^2} \left[\mu_{\mathrm{A}} + 2\frac{e_0}{\eta_0} e_{\mathrm{A}} - \left(\frac{e_0}{\eta_0}\right)^2 \eta_{\mathrm{A}} \right]^2 + \left(\frac{\rho_1}{\rho_0}\right)^2 \right\} \tag{105}$$

Here it is denoted

$$\begin{aligned} \mu_{\mathrm{A}} &= \mu_1 a_{11} + e_1 a_{21} = \frac{2}{\Delta}\left[2\mu_1(\mu_0\eta_0 + e_0^2) + \mu_0(\mu_1\eta_1 + e_1^2)\right] \\ e_{\mathrm{A}} &= \mu_1 a_{12} + e_1 a_{22} = \frac{2}{\Delta}\left[2e_1(\mu_0\eta_0 + e_0^2) + e_0(\mu_1\eta_1 + e_1^2)\right] \\ \eta_{\mathrm{A}} &= \eta_1 a_{22} + e_1 a_{12} = \frac{2}{\Delta}\left[2\eta_1(\mu_0\eta_0 + e_0^2) + \eta_0(\mu_1\eta_1 + e_1^2)\right] \end{aligned} \tag{106}$$

where a_{ij} and Δ are determined in (58) and (60), respectively.

Let us note that quantities μ_{A}, e_{A}, η_{A} were introduced and calculated explicitly for the cylindrical fiber with circular cross-section (Levin et al., 2000). The expressions (106) obtained here coincide completely with those found in the mentioned paper. Besides that, the expression (105) for the total scattering cross-section is in accordance with $Q(\omega)$ obtained by Levin and Michelitsch (2001).

5. Propagation of axial shear waves in the piezoelectric medium reinforced by a random set of fibers

We examine now an unbounded elastic piezoelectric medium with properties $\mathbf{L}^0$ and density ρ_0, containing a spatially homogeneous random set of parallel continuous fibers with properties $\mathbf{L}$, density ρ and having all the same radius a. In the $\mathbf{y}$-plane the cross-sections of the fibers occupy a system of isolated regions S_k with characteristic functions $S_k(\mathbf{y})$, $k = 1, 2, \ldots$ The electroelastic fields in such a medium satisfy the equations similar to (24)

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}^0(\mathbf{y}) + \int \left[\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{L}^1\boldsymbol{F}(\mathbf{y}') + \rho_1\omega^2\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{T}_1\mathbf{f}(\mathbf{y}')\right] S(\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{107}$$

$$\boldsymbol{F}(\mathbf{y}) = \boldsymbol{F}^0(\mathbf{y}) + \int \left[\mathbf{P}(\mathbf{y}-\mathbf{y}')\mathbf{L}^1\boldsymbol{F}(\mathbf{y}') + \rho_1\omega^2\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{T}_1\mathbf{f}(\mathbf{y}')\right] S(\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{108}$$

where $S(\mathbf{y})$ denotes the characteristic function of the region $S = \sum_k S_k$.

In Eqs. (107) and (108) the integral is performed over the region occupied by the inclusions. Thus, the main unknowns of the problem are the fields inside of the fibers.

Let us consider a realization of a random set of fibers in a homogeneous matrix. The distribution of the fiber cross-sections in the transverse plane is assumed to be homogeneous and isotropic. If the incident field is a plane monochromatic wave the mean fields are also plane waves in many important cases. The main difficulty in constructing these fields is the appropriate description of the interaction between many inclusions in composite media. Strictly speaking, in order to construct these fields we have to find the detailed wave fields for every realization of the random set of inclusions and then to average the result over an ensemble of realizations of this set. The principal difficulties of this problem enable us only to find its approximate solution. Self-consistent schemes are powerful tools to obtain such solutions.

Here we use one of the self-consistent schemes named effective field method. This method is based on the following hypothesis (for statics this variant of the effective field method was developed by Kanaun (1983)):

H1 *Every fiber in the composite behaves as an isolated one (with number $k = 1, 2, \ldots$) in the medium (matrix) affected by external fields $\mathbf{f}^*_{(k)}(\mathbf{y})$ and $\boldsymbol{F}^*_{(k)}(\mathbf{y})$. The latter are the sum of the external fields $\mathbf{f}^0(\mathbf{y})$ and $\boldsymbol{F}^0(\mathbf{y})$ applied to the medium and the fields scattered on all surrounding fibers.*

Let now $\mathbf{f}^*(\mathbf{y})$ and $\boldsymbol{F}^*(\mathbf{y})$ be the fields coinciding with $\mathbf{f}^*_{(k)}(\mathbf{y})$ and $\boldsymbol{F}^*_{(k)}(\mathbf{y})$ when $\mathbf{y} \in S_k$. With the help of definition

$$S(\mathbf{y}, \mathbf{y}') = \sum_{i \neq k} S_i(\mathbf{y}'), \quad \mathbf{y} \in S_k \tag{109}$$

we may write for an arbitrary point $\mathbf{y}$ inside of domain S

$$\mathbf{f}^*(\mathbf{y}) = \mathbf{f}^0(\mathbf{y}) + \int \left[\nabla \mathbf{G}(\mathbf{y} - \mathbf{y}')\mathbf{L}^1 \boldsymbol{F}(\mathbf{y}') + \rho_1 \omega^2 \mathbf{G}(\mathbf{y} - \mathbf{y}')\mathbf{T}_1 \mathbf{f}(\mathbf{y}')\right] S(\mathbf{y}, \mathbf{y}')\, \mathrm{d}\mathbf{y}' \tag{110}$$

$$\boldsymbol{F}^*(\mathbf{y}) = \boldsymbol{F}^0(\mathbf{y}) + \int \left[\mathbf{P}(\mathbf{y} - \mathbf{y}')\mathbf{L}^1 \boldsymbol{F}(\mathbf{y}') + \rho_1 \omega^2 \nabla \mathbf{G}(\mathbf{y} - \mathbf{y}')\mathbf{T}_1 \mathbf{f}(\mathbf{y}')\right] S(\mathbf{y}, \mathbf{y}')\, \mathrm{d}\mathbf{y}' \tag{111}$$

Hypothesis H1 reduces the problem of interaction between many inclusions to a one-particle problem. In Section 3 this problem was solved in the long-wave approximation. In this approximation we suppose that the fields $\mathbf{f}^*(\mathbf{y})$ and $\boldsymbol{F}^*(\mathbf{y})$ are constant in every region occupied by the fiber cross-sections (but may vary randomly from one fiber cross-section to another) we can write according to expressions (53) and (56)

$$\begin{aligned}
&\mathbf{f}(\mathbf{x}) = \mathbf{d}\mathbf{f}^*(\mathbf{x}), \quad \boldsymbol{F}(\mathbf{x}) = \mathbf{D}\boldsymbol{F}^*(\mathbf{x}) \\
&\mathbf{d} = \mathbf{d}_\mathrm{R} + \frac{\mathrm{i}\pi}{4}(k_0 a)^2 \frac{\rho_1}{\rho_0}\mathbf{d}_{\boldsymbol{I}}, \quad \mathbf{d}_\mathrm{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{d}_{\boldsymbol{I}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \\
&\mathbf{D} = \mathbf{A} - \frac{\mathrm{i}\pi}{4}(k_0 a)^2 \mathbf{B}
\end{aligned} \tag{112}$$

where $\mathbf{A}$ and $\mathbf{B}$ were determined in (57)–(61).

Substitution of expressions (112) into the right-hand side of Eqs. (107), (108) and (110), (111) allows us to express the electroelastic fields $\mathbf{f}(\mathbf{x})$, $\boldsymbol{F}(\mathbf{x})$ at an arbitrary point of the medium by the local external fields

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}^0(\mathbf{y}) + \int \left[\nabla \mathbf{G}(\mathbf{y} - \mathbf{y}')\mathbf{L}^\mathrm{D} \boldsymbol{F}^*(\mathbf{y}') + \rho_1 \omega^2 \mathbf{G}(\mathbf{y} - \mathbf{y}')\mathbf{d}_1 \mathbf{f}^*(\mathbf{y}')\right] S(\mathbf{y}')\, \mathrm{d}\mathbf{y}' \tag{113}$$

$$\boldsymbol{F}(\mathbf{y}) = \boldsymbol{F}^0(\mathbf{y}) + \int \left[\mathbf{P}(\mathbf{y} - \mathbf{y}')\mathbf{L}^\mathrm{D} \boldsymbol{F}^*(\mathbf{y}') + \rho_1 \omega^2 \nabla \mathbf{G}(\mathbf{y} - \mathbf{y}')\mathbf{d}_1 \mathbf{f}^*(\mathbf{y}')\right] S(\mathbf{y}')\, \mathrm{d}\mathbf{y}' \tag{114}$$

where it is denoted

$$\mathbf{L}^{\mathrm{D}} = \mathbf{L}^1\mathbf{D}, \quad \mathbf{d}_1 = d\mathbf{T}_1, \quad d = 1 + \frac{\rho_1}{\rho_0}\frac{\mathrm{i}\pi}{4}(k_0 a)^2 \tag{115}$$

and obtain a system of self-consistent equations to determine the fields $\mathbf{f}^*(\mathbf{y})$ and $\boldsymbol{F}^*(\mathbf{y})$

$$\mathbf{f}^*(\mathbf{y}) = \mathbf{f}^0(\mathbf{y}) + \int \left[\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\boldsymbol{F}^*(\mathbf{y}') + \rho_1\omega^2\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\mathbf{f}^*(\mathbf{y}')\right]S(\mathbf{y},\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{116}$$

$$\boldsymbol{F}^*(\mathbf{y}) = \boldsymbol{F}^0(\mathbf{y}) + \int \left[\mathbf{P}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\boldsymbol{F}^*(\mathbf{y}') + \rho_1\omega^2\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\mathbf{f}^*(\mathbf{y}')\right]S(\mathbf{y},\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{117}$$

Because we are concerned with a random set of fibers the fields $\mathbf{f}(\mathbf{y})$, $\boldsymbol{F}(\mathbf{y})$ and $\mathbf{f}^*(\mathbf{y})$, $\boldsymbol{F}^*(\mathbf{y})$ are random functions. By taking the ensemble average of both sides of Eqs. (113), (114) we find

$$\langle\mathbf{f}(\mathbf{y})\rangle = \mathbf{f}^0(\mathbf{y}) + p\int \left[\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\widehat{\boldsymbol{F}}^*(\mathbf{y}') + \rho_1\omega^2\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\widehat{\mathbf{f}}^*(\mathbf{y}')\right]\mathrm{d}\mathbf{y}' \tag{118}$$

$$\langle\boldsymbol{F}(\mathbf{y})\rangle = \boldsymbol{F}^0(\mathbf{y}) + p\int \left[\mathbf{P}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\widehat{\boldsymbol{F}}^*(\mathbf{y}') + \rho_1\omega^2\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\widehat{\mathbf{f}}^*(\mathbf{y}')\right]\mathrm{d}\mathbf{y}' \tag{119}$$

where it is denoted

$$\widehat{\boldsymbol{F}}^*(\mathbf{y}) = \langle\boldsymbol{F}(\mathbf{y})|\mathbf{y}\rangle, \quad \widehat{\mathbf{f}}^*(\mathbf{y}) = \langle\mathbf{f}(\mathbf{y})|\mathbf{y}\rangle \tag{120}$$

Symbol $\langle\cdot|\mathbf{y}\rangle$ depicts the ensemble mean under the condition that point $\mathbf{y}$ is located in the region S occupied by the fiber cross-sections, $p = \langle S(\mathbf{y})\rangle$ is the volume concentration of the fibers.

It follows from Eqs. (118), (119) that the average fields $\langle\mathbf{f}(\mathbf{y})\rangle$ and $\langle\boldsymbol{F}(\mathbf{y})\rangle$ at an arbitrary point $\mathbf{y}$ of a transverse plane of composite material are expressed via the conditional means of the effective fields $\widehat{\mathbf{f}}^*(\mathbf{y})$ and $\widehat{\boldsymbol{F}}^*(\mathbf{y})$. These averages can be obtained with the help of Eqs. (116), (117). After the averaging of both parts of these equations under the conditions $\mathbf{y} \in S$ we can write

$$\widehat{\mathbf{f}}^*(\mathbf{y}) = \mathbf{f}^0(\mathbf{y}) + p\int \left[\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\widehat{\boldsymbol{F}}^*(\mathbf{y},\mathbf{y}') + \rho_1\omega^2\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\widehat{\boldsymbol{f}}^{\,*}(\mathbf{y},\mathbf{y}')\right]\Psi(\mathbf{y}-\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{121}$$

$$\widehat{\boldsymbol{F}}^*(\mathbf{y}) = \boldsymbol{F}^0(\mathbf{y}) + p\int \left[\mathbf{P}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\widehat{\boldsymbol{F}}^*(\mathbf{y},\mathbf{y}') + \rho_1\omega^2\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\widehat{\mathbf{f}}^*(\mathbf{y},\mathbf{y}')\right]\Psi(\mathbf{y}-\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{122}$$

where it is denoted

$$\widehat{\boldsymbol{F}}^*(\mathbf{y},\mathbf{y}') = \langle\boldsymbol{F}^*(\mathbf{y}')|\mathbf{y}',\mathbf{y}\rangle, \quad \widehat{\mathbf{f}}^*(\mathbf{y},\mathbf{y}') = \langle\mathbf{f}^*(\mathbf{y}')|\mathbf{y}',\mathbf{y}\rangle \quad \Psi(\mathbf{y}-\mathbf{y}') = \frac{\langle S(\mathbf{y},\mathbf{y}')|\mathbf{y}\rangle}{\langle S(\mathbf{y})\rangle} \tag{123}$$

In these expressions $\langle\cdot|\mathbf{y},\mathbf{y}'\rangle$ is the mean under the condition $\mathbf{y},\mathbf{y}' \in S$ and the following relationship was taken into account

$$\begin{aligned}\left\langle\mathbf{f}^*(\mathbf{y}')S(\mathbf{y},\mathbf{y}')\right\rangle &= \left\langle\mathbf{f}^*(\mathbf{y}')|\mathbf{y}',\mathbf{y}\right\rangle\left\langle S(\mathbf{y},\mathbf{y}')|\mathbf{y}\right\rangle \\ \left\langle\boldsymbol{F}^*(\mathbf{y}')S(\mathbf{y},\mathbf{y}')\right\rangle &= \left\langle\boldsymbol{F}^*(\mathbf{y}')|\mathbf{y}',\mathbf{y}\right\rangle\left\langle S(\mathbf{y},\mathbf{y}')|\mathbf{y}\right\rangle\end{aligned} \tag{124}$$

This result follows from the definition of the conditional means of random functions $\mathbf{f}^*(\mathbf{y})$ and $\boldsymbol{F}^*(\mathbf{y})$.

In general, mean $\langle\cdot|\mathbf{y}',\mathbf{y}\rangle$ differs from $\langle\cdot|\mathbf{y}\rangle$. To obtain the expressions for the means $\widehat{\mathbf{f}}^*(\mathbf{y},\mathbf{y}')$ and $\boldsymbol{F}^*(\mathbf{y},\mathbf{y}')$ we can average both sides of Eqs. (116), (117) under the condition $\mathbf{y},\mathbf{y}' \in S$. But in the right-hand sides of these equations we will have the conditional means of more complex structure. Thus, we obtain a hierarchy of equations connecting the conditional means of the effective fields $\mathbf{f}^*(\mathbf{y})$ and $\boldsymbol{F}^*(\mathbf{y})$. To close this hierarchy we must invoke certain additional assumptions concerning the statistical properties of the effective fields. The simplest assumption is represented by the analogue of the so-called "quasicrystalline approximation" proposed by Lax (1951, 1952), according to which the means $\langle\cdot|\mathbf{y}',\mathbf{y}\rangle$ and $\langle\cdot|\mathbf{y}\rangle$ coincide. This results in

$$\widehat{\mathbf{f}}^*(\mathbf{y}) = \mathbf{f}^0(\mathbf{y}) + p\int\left[\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\widehat{\boldsymbol{F}}^*(\mathbf{y}') + \rho_1\omega^2\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\widehat{\mathbf{f}}^*(\mathbf{y}')\right]\Psi(\mathbf{y}-\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{125}$$

$$\widehat{F}^*(\mathbf{y}) = \boldsymbol{F}^0(\mathbf{y}) + p\int\left[\mathbf{P}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\widehat{\boldsymbol{F}}^*(\mathbf{y}') + \rho_1\omega^2\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\widehat{\mathbf{f}}^*(\mathbf{y}')\right]\Psi(\mathbf{y}-\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{126}$$

It follows from the definition of function $S(\mathbf{y},\mathbf{y}')$ (109) that $\Psi(\mathbf{y})$ is a continuous function and

$$\Psi(\mathbf{y}) = 0, \quad \text{when} \quad \mathbf{y} = 0 \tag{127}$$

Because of weakening in geometrical linkage between the position of the fibers when the distances between them increase, the following relation holds

$$\Psi(\mathbf{y}) \to 1, \quad \text{when} \quad |\mathbf{y}| \to \infty \tag{128}$$

Function $\Psi(\mathbf{y})$ defines the shape of the "correlation hole" inside which a typical fiber is located.

If the random set of fiber cross-sections possesses some symmetry (in the statistical sense) it affects the symmetry of function $\Psi(\mathbf{y})$. Particularly, if the random set of cross-sections is isotropic, function $\Psi(\mathbf{y})$ depends only on $|\mathbf{y}|$ i.e. $\Psi(\mathbf{y}) = \Psi(|\mathbf{y}|)$. The deviation of the random set of fibers from isotropy can mean the existence of texture. Let us assume that there exists a linear transformation of **y**-plane that rearranges function $\Psi(\mathbf{y})$ into a spatially symmetric one

$$\mathbf{z} = \boldsymbol{\alpha}(\mathbf{a})\mathbf{y}, \quad \Psi(\boldsymbol{\alpha}^{-1}(\mathbf{a})\mathbf{y}) = \Psi(|\mathbf{y}|) \tag{129}$$

In this case an ellipse defined by the expression

$$|\boldsymbol{\alpha}(\mathbf{a})\mathbf{y}| = 1$$

with semi-axes $\mathbf{a} = (a_1, a_2)$ describes the shape of the correlation hole (corresponding to texture). In the general case it is impossible to find such a transformation.

Eliminating the external fields $\mathbf{f}^0(\mathbf{y})$ and $\boldsymbol{F}^0(\mathbf{y})$ from Eqs. (118), (119) and (125), (126), we get a system of equations which couple the effective fields $\widehat{\mathbf{f}}^*(\mathbf{y})$, $\widehat{\boldsymbol{F}}^*(\mathbf{y})$ and average fields $\langle\mathbf{f}(\mathbf{y})\rangle$, $\langle\boldsymbol{F}(\mathbf{y})\rangle$ in the composite material

$$\widehat{\mathbf{f}}^*(\mathbf{y}) = \langle\mathbf{f}(\mathbf{y})\rangle - p\int\left[\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\widehat{\boldsymbol{F}}^*(\mathbf{y}') + \rho_1\omega^2\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\widehat{\mathbf{f}}^*(\mathbf{y}')\right]\Phi(\mathbf{y}-\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{130}$$

$$\widehat{\boldsymbol{F}}^*(\mathbf{y}) = \langle\boldsymbol{F}(\mathbf{y})\rangle - p\int\left[\mathbf{P}(\mathbf{y}-\mathbf{y}')\mathbf{L}^{\mathrm{D}}\widehat{\boldsymbol{F}}^*(\mathbf{y}') + \rho_1\omega^2\nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\mathbf{d}_1\widehat{\mathbf{f}}^*(\mathbf{y}')\right]\Phi(\mathbf{y}-\mathbf{y}')\,\mathrm{d}\mathbf{y}' \tag{131}$$

where it is denoted

$$\Phi(\mathbf{y}) = 1 - \Psi(\mathbf{y}) \tag{132}$$

For a spatially homogeneous random set of fibers $\Phi(\mathbf{y})$ is a smooth function which quickly goes to zero outside a region having a size of the order of the correlation hole size. In the long-wave approximation we can neglect the change of the fields $\widehat{\mathbf{f}}^*(\mathbf{y}), \widehat{\boldsymbol{F}}^*(\mathbf{y})$ in this region. If we assume that the distribution of the fiber cross-sections is isotropic $\Phi(\mathbf{y}) = \Phi(|\mathbf{y}|)$ Eqs. (130) and (131) take the form

$$\widehat{\mathbf{f}}^*(\mathbf{y}) = \langle\mathbf{f}(\mathbf{y})\rangle - p\rho_1\omega^2\mathbf{G}^{\Phi}\mathbf{d}_1\widehat{\mathbf{f}}^*(\mathbf{y}) \tag{133}$$

$$\widehat{\boldsymbol{F}}^*(\mathbf{y}) = \langle\boldsymbol{F}(\mathbf{y})\rangle - p\mathbf{P}^{\Phi}\mathbf{L}^{\mathrm{D}}\widehat{\boldsymbol{F}}^*(\mathbf{y}) \tag{134}$$

where it is denoted

$$\mathbf{G}^{\Phi} = \int \mathbf{G}(\mathbf{y})\Phi(|\mathbf{y}|)\,\mathrm{d}\mathbf{y}, \quad \mathbf{P}^{\Phi} = \int \mathbf{P}(\mathbf{y})\Phi(|\mathbf{y}|)\,\mathrm{d}\mathbf{y}, \tag{135}$$

and the relation

$$\int \nabla\mathbf{G}(\mathbf{y})\Phi(|\mathbf{y}|)\,\mathrm{d}\mathbf{y} = 0 \tag{136}$$

is taken into account.

System (133) is equivalent to the two equations

$$\begin{aligned} \widehat{u}^*(\mathbf{y}) &= \langle u(\mathbf{y})\rangle - p\frac{\rho_1}{\rho_0}k_0^2 G^{\Phi} d\widehat{u}^*(\mathbf{y}) \\ \widehat{\varphi}^*(\mathbf{y}) &= \langle \varphi(\mathbf{y})\rangle - p\frac{\rho_1}{\rho_0}k_0^2 G^{\Phi} d\frac{e_0}{\eta_0}\widehat{u}^*(\mathbf{y}) \end{aligned} \tag{137}$$

where

$$G^{\Phi} = \frac{\mathrm{i}\pi}{2}\int_0^{\infty} H_0(k_0 r)\Phi(r) r\,\mathrm{d}r \tag{138}$$

Because of the mentioned properties of the function $\Phi(r)$ we can represent the function $H_0(k_0 r)$ in this integral by the main terms of its asymptotic expansion, in the long-wave limit $k_0 r \ll 1$ where l is the correlation hole radius (a quantity having the order of the mean distances between the fiber cross-sections in the **y**-plane)

$$\frac{\mathrm{i}\pi}{2}H_0^{(1)}(k_0 r) \sim -\left(\ln(k_0 r) - \frac{\mathrm{i}\pi}{2}\right) \tag{139}$$

If we take into account (as in Section 3) only the main terms in the real and imaginary parts of Eq. (137) we can write

$$\begin{aligned} \widehat{u}^*(\mathbf{y}) &= \left[1 - p\frac{\rho_1}{\rho_0}\frac{\mathrm{i}\pi}{4}(k_0 a)^2 J\right]\langle u(\mathbf{y})\rangle \\ \widehat{\varphi}^*(\mathbf{y}) &= \langle \varphi(\mathbf{y})\rangle - p\frac{\rho_1}{\rho_0}\frac{\mathrm{i}\pi}{4}(k_0 a)^2 J\frac{e_0}{\eta_0}\langle u(\mathbf{y})\rangle \end{aligned} \tag{140}$$

where it is denoted

$$J = \frac{2}{a^2}\int_0^{\infty}\Phi(r) r\,\mathrm{d}r \tag{141}$$

Let us return now to Eq. (135). It is convenient to write operator $\mathbf{P}^{\Phi}$ in the form

$$\begin{aligned} \mathbf{P}^{\Phi} &= \int \mathbf{P}(\mathbf{y})\Phi(|\mathbf{y}|)\,\mathrm{d}\mathbf{y} = \int\left[\frac{1}{(2\pi)^2}\int \mathbf{P}(\mathbf{k})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{y}}\,\mathrm{d}\mathbf{k}\right]\Phi(|\mathbf{y}|)\,\mathrm{d}\mathbf{y} \\ &= \frac{1}{2\pi}\int_0^{\infty}\left[\int \mathbf{P}(\mathbf{k})J_0(kr)\,\mathrm{d}\mathbf{k}\right]\Phi(r) r\,\mathrm{d}r \end{aligned} \tag{142}$$

where

$$\mathbf{P}(\mathbf{k}) = \left[\frac{1}{\eta_0}\mathbf{T}_2 - \frac{1}{\widehat{\mu}_0}\left(1 + \frac{k_0^2}{k^2 - k_0^2}\right)\mathbf{T}_3\right] n_i n_j, \quad n_i = \frac{k_i}{k} \tag{143}$$

The internal integral in (142) can be transformed into one-dimensional integrals in the following way

$$\int \mathbf{P}(\mathbf{k})J_0(kr)\,\mathrm{d}\mathbf{k} = \int_0^\infty J_0(kr)k\,\mathrm{d}k \int_0^{2\pi} \mathbf{P}(\mathbf{k})\,\mathrm{d}\phi$$
$$= \pi\left[\left(\frac{1}{\eta_0}\mathbf{T}_2 - \frac{1}{\widehat{\mu}_0}\mathbf{T}_3\right)\int_0^\infty J_0(kr)k\,\mathrm{d}k - \left(k_0\int_0^\infty \frac{J_0(kr)k\,\mathrm{d}k}{k^2-k_0^2}\right)\mathbf{T}_3\right]\theta_{ij} \tag{144}$$

Taking into account the relations

$$\int_0^\infty J_0(kr)k\,\mathrm{d}k = \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}}\,\mathrm{d}\mathbf{k} = 2\pi\delta(\mathbf{y})$$
$$2\pi\int_0^\infty \delta(\mathbf{y})\Phi(r)r\,\mathrm{d}r = \int \delta(\mathbf{y})\Phi(r)\,\mathrm{d}\mathbf{y} = \Phi(0) = 1 \tag{145}$$
$$\int_0^\infty \frac{J_0(kr)k\,\mathrm{d}k}{k^2-k_0^2} = \frac{\mathrm{i}\pi}{2}H_0^{(1)}(k_0r)$$

where the last integral is derived in detail in Appendix A. We obtain finally

$$\mathbf{P}^{\Phi} = \mathbf{P}^{\mathrm{R}} - \frac{\mathrm{i}\pi}{4}(k_0a)^2 J\mathbf{P}^{I} \tag{146}$$

where operators $\mathbf{P}^{\mathrm{R}}$ and $\mathbf{P}^{I}$ are determined in (55).

The solution of Eq. (134) with respect to $\widehat{\boldsymbol{F}}^{*}(\mathbf{y})$ and prescribed accuracy can be written in the form

$$\widehat{\boldsymbol{F}}^{*}(\mathbf{y}) = \mathbf{D}^{\mathrm{R}}\left[\mathbf{I} + p\frac{\mathrm{i}\pi}{4}(k_0a)^2(\mathbf{P}^{\mathrm{R}}\mathbf{L}^{\mathrm{A}} + J\mathbf{I})\mathbf{P}^{I}\mathbf{L}^{\mathrm{R}}\right]\langle \boldsymbol{F}(\mathbf{y})\rangle$$
$$\mathbf{D}^{\mathrm{R}} = \left(\mathbf{I} + p\mathbf{P}^{\mathrm{R}}\mathbf{L}^{\mathrm{A}}\right)^{-1}, \quad \mathbf{L}^{\mathrm{A}} = \mathbf{L}^{1}\mathbf{A}, \quad \mathbf{L}^{\mathrm{R}} = \mathbf{L}^{\mathrm{A}}\mathbf{D}^{\mathrm{R}} \tag{147}$$

Substitution of obtained expressions (140) and (147) in the right-hand side of Eq. (118) gives after some lengthy but elementary algebra

$$\langle \mathbf{f}(\mathbf{y})\rangle = \mathbf{f}^0(\mathbf{y}) + p\int \nabla\mathbf{G}(\mathbf{y}-\mathbf{y}')\left[\mathbf{L}^{\mathrm{R}} - \frac{\mathrm{i}\pi}{4}(k_0a)^2(1-pJ)\mathbf{L}^{\mathrm{R}}\mathbf{P}^{I}\mathbf{L}^{\mathrm{R}}\right]\langle \boldsymbol{F}(\mathbf{y}')\rangle\,\mathrm{d}\mathbf{y}'$$
$$+ p\rho_1\omega^2\int \mathbf{G}(\mathbf{y}-\mathbf{y}')\left[1 + \frac{\mathrm{i}\pi}{4}(k_0a)^2(1-pJ)\frac{\rho_1}{\rho_0}\right]\mathbf{T}_1\langle \mathbf{f}(\mathbf{y}')\rangle\,\mathrm{d}\mathbf{y}' \tag{148}$$

Let us apply to the both sides of Eq. (148) operator $\nabla\mathbf{L}^0\nabla + \rho_0\omega^2\mathbf{T}_1$. Taking into account that Green's function $\mathbf{G}(\mathbf{y})$ and incident fields $\mathbf{f}^0(\mathbf{y})$ satisfy the equations

$$\left(\nabla\mathbf{L}^0\nabla + \rho_0\omega^2\mathbf{T}_1\right)\mathbf{G}(\mathbf{y}) = -\mathbf{I}\delta(\mathbf{y}), \quad (\nabla\mathbf{L}^0\nabla + \rho_0\omega^2\mathbf{T}_1)\mathbf{f}^0(\mathbf{y}) = 0 \tag{149}$$

we obtain an equation determining $\langle \mathbf{f}(\mathbf{y})\rangle$ in the form

$$\left(\nabla\mathbf{L}^{*}\nabla + \rho_{*}\omega^2\mathbf{T}_1\right)\langle \mathbf{f}(\mathbf{y})\rangle = 0 \tag{150}$$

where it is denoted

$$\mathbf{L}^{*} = \mathbf{L}^{\mathrm{s}} - p\frac{\mathrm{i}\pi}{4}(k_0a)^2(1-pJ)\mathbf{L}^{I}, \quad \mathbf{L}^{\mathrm{s}} = \mathbf{L}^0 + p\mathbf{L}^{\mathrm{R}}, \quad \mathbf{L}^{I} = \mathbf{L}^{\mathrm{R}}\mathbf{P}^{I}\mathbf{L}^{\mathrm{R}}$$
$$\rho^{*} = \rho_{\mathrm{s}} + p\frac{\mathrm{i}\pi}{4}(k_0a)^2(1-pJ)\frac{\rho_1^2}{\rho_0}, \quad \rho_{\mathrm{s}} = \rho_0 + p\rho_1 \tag{151}$$

Operator $\nabla\mathbf{L}^{*}\nabla + \rho_{*}\omega^2\mathbf{T}_1$ can be called *effective wave operator* that describes the axial shear wave propagation in the piezoelectric medium reinforced by unidirectional aligned continuous fibers. It has the same form that has the analogous operator for homogeneous material with electroelastic characteristics $\mathbf{L}^*$ and density ρ_* which are yet complex quantities. Their real parts determine the velocity of the axial shear wave propagation and their imaginary parts determine the attenuation factor for these waves. Let us note that the real parts of all these characteristics are independent of frequency. It was to be expected in the long-wave approximation because then all terms depending on frequency in the real parts of all expressions are neglected in comparison with the constant ("static") parts. In result we obtain the effective wave operator which describes the wave propagation in the medium with wave attenuation but without dispersion.

The operator of "static" electroelastic characteristics $\mathbf{L}^s$ in details can be written in the form

$$\mathbf{L}^s = \begin{pmatrix} \mu_s & e_s \\ e_s & -\eta_s \end{pmatrix} \tag{152}$$

$$\mu_s = \mu_0 + p\mu_R, \quad e_s = e_0 + pe_R, \quad \eta_s = \eta_0 + p\eta_R \tag{153}$$

where it is denoted

$$\begin{aligned} \mu_R &= \frac{1}{\Delta}\left[\mu_1 + (1-p)\mu_0 \frac{\mu_1\eta_1 + e_1^2}{2(\mu_0\eta_0 + e_0^2)}\right] \\ e_R &= \frac{1}{\Delta}\left[e_1 + (1-p)e_0 \frac{\mu_1\eta_1 + e_1^2}{2(\mu_0\eta_0 + e_0^2)}\right] \\ \eta_R &= \frac{1}{\Delta}\left[\eta_1 + (1-p)\eta_0 \frac{\mu_1\eta_1 + e_1^2}{2(\mu_0\eta_0 + e_0^2)}\right] \\ \Delta &= \left[1 + (1-p)\frac{\eta_1\mu_0 + e_1e_0}{2(\mu_0\eta_0 + e_0^2)}\right]\left[1 + (1-p)\frac{\mu_1\eta_0 + e_1e_0}{2(\mu_0\eta_0 + e_0^2)}\right] - \frac{(1-p)^2}{4(\mu_0\eta_0 + e_0^2)^2}(e_1\eta_0 - \eta_1e_0)(\mu_1e_0 - e_1\mu_0) \end{aligned} \tag{154}$$

Expressions (152), (154) coincide with those obtained by Levin (1996b).

Operator $\mathbf{L}^I$ has the analogous form

$$\begin{aligned} \mathbf{L}^I &= \begin{pmatrix} \mu_I & e_I \\ e_I & -\eta_I \end{pmatrix} \\ \mu_I &= \frac{1}{2\overline{\mu}_0}\left(\mu_R + \frac{e_0}{\eta_0}e_R\right)^2 \\ e_I &= \frac{1}{2\overline{\mu}_0}\left(\mu_R + \frac{e_0}{\eta_0}e_R\right)\left(e_R - \frac{e_0}{\eta_0}\eta_R\right) \\ \eta_I &= -\frac{1}{2\overline{\mu}_0}\left(e_R + \frac{e_0}{\eta_0}\eta_R\right)^2 \end{aligned} \tag{155}$$

6. Effective wave velocity and attenuation factor

The equation of motion can now be written as

$$\begin{aligned} &\mu_*\Delta\langle u(\mathbf{y})\rangle + e_*\Delta\langle\varphi(\mathbf{y})\rangle + \rho_*\omega^2\langle u(\mathbf{y})\rangle = 0 \\ &e_*\Delta\langle u(\mathbf{y})\rangle - \eta_*\Delta\langle\varphi\rangle = 0 \end{aligned} \tag{156}$$

Let the average elastic displacement and electric potential be a plane axial shear wave with polarization U and Φ and effective wave number k_*

$$\langle u(\mathbf{y})\rangle = U\mathrm{e}^{\mathrm{i}k_*\mathbf{n}\cdot\mathbf{y}}, \quad \langle \varphi(\mathbf{y})\rangle = \Phi\mathrm{e}^{\mathrm{i}k_*\mathbf{n}\cdot\mathbf{y}} \tag{157}$$

It follows from the second Eq. (156) that

$$\Phi = \frac{e_*}{\eta_*}U \tag{158}$$

and the first Eq. (156) gives

$$\left(\mu_* + \frac{e_*^2}{\eta_*}\right)k_*^2 - \rho_*\omega^2 = 0 \tag{159}$$

Taking into account that in the long-wave approximation ($k_0 a \ll 1$)

$$\mu_* + \frac{e_*^2}{\eta_*} = \overline{\mu}_\mathrm{s} - p\frac{\mathrm{i}\pi}{4}(k_0 a)^2(1-pJ)\left[\mu_{\boldsymbol{I}} + 2\frac{e_\mathrm{s}}{\eta_\mathrm{s}}e_{\boldsymbol{I}} - \left(\frac{e_\mathrm{s}}{\eta_\mathrm{s}}\right)^2\eta_{\boldsymbol{I}}\right] = \overline{\mu}_\mathrm{s} - p\frac{\mathrm{i}\pi}{4}(k_0 a)^2(1-pJ)\frac{\widehat{\mu}_{\boldsymbol{I}}^2}{2\widehat{\mu}_0},$$
$$\overline{\mu}_\mathrm{s} = \mu_\mathrm{s} + \frac{e_\mathrm{s}^2}{\eta_\mathrm{s}}, \quad \overline{\mu}_{\boldsymbol{I}} = \mu_\mathrm{R} + \frac{e_0}{\eta_0}e_\mathrm{R} + \frac{e_\mathrm{s}}{\eta_\mathrm{s}}\left(e_\mathrm{R} - \frac{e_0}{\eta_0}\eta_\mathrm{R}\right) \tag{160}$$

and solving Eq. (159) with respect to k_* we obtain

$$k_* = k_\mathrm{s} + \mathrm{i}\gamma, \quad k_\mathrm{s} = \frac{\omega}{v_\mathrm{s}}, \quad v_\mathrm{s} = \sqrt{\frac{\overline{\mu}_\mathrm{s}}{\rho_\mathrm{s}}} \tag{161}$$

Here v_s is the wave velocity in the composite material, γ is the attenuation factor that is determined by the expression

$$\gamma = \frac{1}{16}\frac{n_0\pi^2}{\rho_0\rho_\mathrm{s}}(1-pJ)(k_\mathrm{s}a)^3 a\left[\frac{\overline{\mu}_{\boldsymbol{I}}^2}{v_0^4} + 2\left(\rho_1\frac{k_0}{k_\mathrm{s}}\right)^2\right], \quad v_0^2 = \frac{\overline{\mu}_0}{\rho_0} \tag{162}$$

where n_0 is the numerical concentration of the fibers ($p = n_0\pi a^2$).

In accordance with its physical meaning the attenuation factor has to be a positive quantity. Consequently the multiplier $1 - pJ$ in (162) should satisfy the condition

$$1 - \frac{2p}{a^2}\int_0^\infty \Phi(r)r\,\mathrm{d}r \geqslant 0 \tag{163}$$

This imposes a constraint on the fiber volume concentration p for which the resulting formula (162) remains physically consistent. For example, for a function (“well-stirred” approximation)

$$\Phi(r) = 1 - H\left(\frac{r}{a} - 2\right) \tag{164}$$

where $H(z)$ is the Heaviside function, γ is positive only for $p \leqslant 0.25$. It follows from here that the expression for γ (162) (in contrast with the expression for the effective wave velocity) is very sensitive to the form of pair correlation function. We must emphasize that the closure condition used above, which determines the structure of multiplier $1 - pJ$ are hardly valid for high fiber concentrations. It is obvious that the correlation in spatial location of the fiber cross-sections increases with increasing p. Hence, sufficiently simple approximations of function $\Phi(r)$ are possible only for a small p. A construction of this function for high fiber concentration presents considerable difficulties (see the discussion of this question, for example, Talbot and Willis (1983)).

Let the fiber cross-sections compose a regular lattice in a homogeneous matrix. In this case the integral from function $\Phi(r)$ equals to the square of the periodic cell and multiplier $1 - pJ$ equals to zero. This corresponds to the well-known fact that a long wave propagation through periodical structures is free from attenuation.

We suppose in conclusion that the volume concentration of fibers is small ($p \ll 1$). In this case we can drop all terms in the right-hand side of (162) having the order O(p). It gives

$$
\begin{aligned}
& e_s, \eta_s \to e_0, \eta_0, \quad \mu_R, e_R, \eta_R \to \mu_A, e_A, \eta_A, \\
& \overline{\mu}_{\boldsymbol{I}} \to \mu_A + 2\frac{e_0}{\eta_0} e_A - \left(\frac{e_0}{\eta_0}\right)^2 \eta_A
\end{aligned}
\tag{165}
$$

and the attenuation factor is determined by the expected expression

$$
\gamma = \frac{1}{2} n_0 Q(\omega) \tag{166}
$$

where $Q(\omega)$ is the total scattering cross-section (105).

7. Numerical example

Here we consider the dependency of the static part of the electroelastic moduli of the volume concentration p of the fibers (Eqs. (152), (154)) for $BaTiO_3$-fibers which are embedded in a PZT-5H-matrix. The p dependency of expressions (152), (154) is plotted in Figs. 2 and 3, respectively. The electroelastic matrix- and fiber moduli which have been used are taken from Huang and Kuo (1996):

PZT-5H-matrix: $\mu_0 = C_{44}^0 = 35.5$ GPa, $\quad e_0 = e_{15}^0 = 17.0C$ m^{-2}
$BaTiO_3$-fibers: $\mu_0 + \mu_1 = C_{44}^0 + C_{44}^1 = 43$ GPa, $\quad e_0 + e_1 = e_{15}^0 + e_{15}^1 = 11.6C$ m^{-2}

Fig. 2 shows the fiber concentration dependency of effective electroelastic moduli starting at $p = 0$ (absence of fibers) to $p = 1$ (limiting case of pure fiber material). We observe only a smooth dependency of $\mu_s = C_{44}$ and $e_s = e_{15}$ of the fiber concentration p.

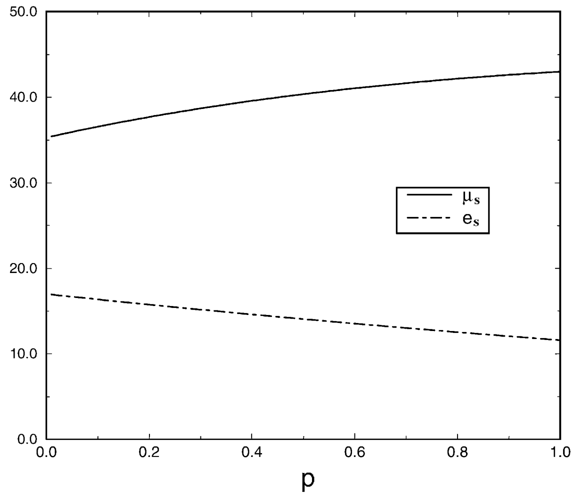

Fig. 2. μ_s (GPa) and e_s (C/m^2) from Eqs. (153) vs. fiber concentration p.

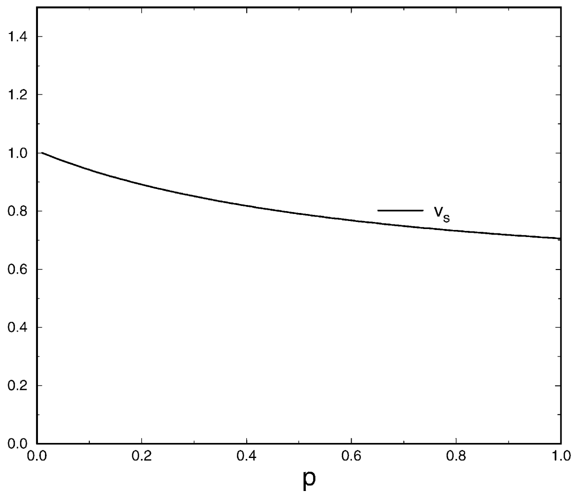

Fig. 3. Normalized propagation velocity $v_s(p)/v_s(0)$ for $\rho = 2\rho_0$, $\rho_0 = \rho_1$.

In Fig. 3 the effective propagation velocity of electroacoustic waves is drawn vs. fiber concentration (Eq. (161)). In this plot we have assumed $\rho_0 = \rho_1$, that is, the mass density of the fiber material is twice as much as the density of matrix material ($\rho = 2\rho_0 = \rho_0 + \rho_1$). We then observe a decrease of the effective wave propagation velocity with increase of fiber volume concentration p. This decrease is caused only by the difference in mass densities of fiber and matrix material determined by Eq. (161) together with (151). This decrease is removed when the mass densities of fiber and matrix material are coincident. Then the effective wave velocity is determined only by the increase of $\sqrt{\mu_s}$ corresponding to Fig. 2.

8. Conclusions

The goal of this paper was the study of effective dynamic characteristics of piezocomposites. In the first part of the paper we considered the scattering problem of electroacoustic axial shear waves for an *isolated* continuous fiber in the framework of the long-wave approximation (Sections 2, 3). In this approximation the set of self-consistent integral equations for the scattered electroelastic field (Eqs. (19), (22)) is reduced to a system of *algebraic* equations (Eqs. (37), (38)). The solution of this equation system requires the explicit evaluation of integrals occurring in its coefficients (Eq. (39)). Crucial for the explicit derivation of these integrals [5] resulting in Eqs. (46) and (47), is the availability of the explicit form of the dynamic electroelastic quasiplane Green's function (Eq. (34)) which was already derived earlier (Levin and Michelitsch, 1999). [6] By using these results for the scattered electroelastic field we derived the total cross-section in explicit form (see Eqs. (100)–(106)) by utilizing the electroacoustic analogue of optical theorem and the long-wave approximation.

Based on these results for the scattering problem of *one isolated fiber* ("one-particle problem") we considered in the second part of the paper (Sections 5, 6) the "*multiple-particle*" *scattering problem* of a *statistical ensemble* of continuous fibers. By introducing statistical assumptions (Eqs. (127)–(129)) characterizing the texture of the fiber distribution, the set of integral equations for the scattered fields on a

[5] For detailed derivations and physical interpretation of these integrals, see Appendix B and C.

[6] A detailed derivation of this Green's function is also given in Appendix A.

random set of fibers (Eqs. (107), (108)) was reduced to a self-consistent scheme, of an effective "one-particle" scattering problem (Eqs. (133), (134)) in the long-wave approximation. In the framework of this effective field method, a wave equation (Eq. (150)) was derived for the average field with an effective wave operator, leading to the effective dynamic characteristics (151) which are complex quantities. From these dynamic characteristics, the effective wave vector was determined (Eq. (161)). Its real part determines the effective wave velocity (Eq. (161)) and its imaginary parts determines the attenuation factor (Eq. (162)) of the considered electroacoustic axial shear wave. All these results for the effective dynamic characteristics were derived in full explicit form in the framework of the long-wave approximation.

As a consequence of the long-wave approximation, the results for the dynamic characteristics of Section 6 cover the lowest orders in their frequency dependencies and thus describe attenuation but not dispersion effects of electroacoustic axial shear waves. The description of dispersion effects requires an approach beyond the long-wave approximation and is therefore highly desirable. The achieved results may inspire further work in this direction and in general for the modelling of dynamic characteristics of fiber reinforced piezocomposites.

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Appendix A

Here we derive the quasiplane dynamic Green's function (34) in the space-frequency and in the *causal* space-time representation, respectively. The problem to obtain this Green's function is reduced in finding scalar Green's functions $\bar{g}(r,t)$ and $\bar{h}(r,t)$, [7] the first being Green's function of scalar wave equation of the form

$$\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}+\epsilon\right)^2-\Delta\right]\bar{g}(r,t)=\delta(t)\delta^2(\mathbf{r}) \tag{A.1}$$

and the second one is Green's function defined by

$$\left[\frac{1}{c^2}\left(\frac{\partial}{\partial t}+\epsilon\right)^2-\Delta\right]\left[\frac{\partial}{\partial t}+\epsilon\right]^2\bar{h}(r,t)=\delta(t)\delta^2(\mathbf{r}) \tag{A.2}$$

In (A.1) and (A.2) we introduced an infinitesimal damping term $\epsilon\to 0+$ leading to causal behavior, that is, $\bar{g}$ and $\bar{h}$ are only non-zero for $t>0$. We observe that Green's function $\bar{h}$ and $\bar{g}$ are related to each other by the convolution

$$\bar{h}(r,t)=\int_{-\infty}^{\infty}f(t-\tau)\bar{g}(r,\tau)\,\mathrm{d}\tau \tag{A.3}$$

where $f(t)$ is the Green's function defined by

$$\left[\frac{\partial}{\partial t}+\epsilon\right]^2 f(t)=\delta(t) \tag{A.4}$$

[7] $\mathbf{r}=(x,y)$ denotes here the plane space vector.

which has the solution $f(t) = e^{-\epsilon t} t \Theta(t)$. Here we introduced the Heaviside step function $\Theta(t)$ which indicates causality and is characterized by $\Theta(t) = 1$ if $t > 0$ and $\Theta(t) = 0$ if $t < 0$. Thus we can obtain $\bar{h}(r, t)$ easily from (A.3) after having determined $\bar{g}(r, t)$. [8] First of all, we obtain by using the residue theorem

$$\bar{g}(k, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}\, d\omega}{k^2 + \left(\epsilon - i\frac{\omega}{c}\right)^2} = c^2 \Theta(t) \frac{\sin(ckt)}{ck} e^{-\epsilon t} \tag{A.5}$$

Here the damping constant $\epsilon \to 0+$ infinitesimally shifts the singularities of the integrand into the *lower complex ω-plane* at $\omega_{1,2} = -i\epsilon \pm ck$. Moreover, to apply the residue theorem we have to close the integration path by semi-circles with radii $|\omega| \to \infty$ in the complex ω-plane. In view of the exponent in (A.5) we have to distinguish the cases $t < 0$ and $t > 0$, respectively. For $t < 0$ we have to close the contour by a semi-circle in the upper complex plane ($\mathrm{Im}\,\omega > 0$) and for $t < 0$ we have to close the contour by a semi-circle in the lower complex plane ($\mathrm{Im}\,\omega < 0$). The contribution of the added integrations over the semi-circles tend to zero when $|\omega| \to \infty$.

For $t < 0$ integral (A.5) vanishes since it has no residues in the upper complex plane. For $t > 0$ the integral (A.5) is non-zero and yields $-2\pi i$ times the sum of the residues located in the lower complex plane at $\omega_{1,2} = -i\epsilon \pm ck$. [9] This property is expressed by the Heaviside $\Theta(t)$-step-function and indicates *causality* and is solely effected by the infinitesimal damping constant ϵ. Expression (A.5) is known as the causal Green's function of an infinitesimally damped harmonic oscillator with eigenfrequency ck fulfilling the differential equation [10]

$$\left[\left(\frac{d}{dt} + \epsilon\right)^2 + c^2 k^2\right] g(k, t) = c^2 \delta(t) \tag{A.6}$$

The exponential term in (A.5) $e^{-\epsilon t} \to 1$ and can therefore be omitted for finite t when $\epsilon \to 0+$. The space time representation is obtained by

$$\bar{g}(r, t) = \frac{1}{(2\pi)^2} \int e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{g}(\mathbf{k}, t)\, d^2k \tag{A.7}$$

This expression we can transform with (A.5) into

$$\bar{g}(r, t) = \Theta(t) \frac{c}{(2\pi)^2} \int_0^{2\pi} d\varphi \int_0^{\infty} \sin(k[ct + kr\cos\varphi])\, dk \tag{A.8}$$

In this expression we have used the property

$$\int_0^{2\pi} d\varphi \sin(kr\cos\varphi) = 0 \tag{A.9}$$

To define the k-integral (A.8) we regularize it as follows

$$\int_0^{\infty} \sin k\lambda\, dk = \lim_{\epsilon \to 0+} \int_0^{\infty} e^{-\epsilon k} \sin k\lambda\, dk = \lim_{\epsilon \to 0+} \mathrm{Re} \frac{1}{\lambda + i\epsilon} \tag{A.10}$$

so that we can write for (A.8)

[8] To determine $\bar{g}(r, \omega)$, it is more convenient to calculate first $\bar{g}(r, t)$.

[9] The negative sign comes into play because the residues are circulated clockwise.

[10] See textbooks of theoretical physics, e.g. Haake (1983).

$$\bar{g}(r,t) = \frac{c\Theta(t)}{(2\pi)^2}\mathrm{Re}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{r\cos\varphi + ct + \mathrm{i}\epsilon} \tag{A.11}$$

To evaluate (A.11) we put $ct/r = \cosh\phi$ where ϕ is real for $ct/r > 1$ and ϕ imaginary for $ct/r < 1$. Then we can write (A.11) in the form

$$\begin{aligned}\bar{g}(r,t) &= +\frac{c\Theta(t)}{(2\pi)^2}\frac{2}{r}\mathrm{Re}\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{i}\varphi}\mathrm{d}\varphi}{\mathrm{e}^{2\mathrm{i}\varphi}+2\,\mathrm{e}^{\mathrm{i}\varphi}\cosh\phi+1} \\ &= +\frac{c\Theta(t)}{(2\pi)^2}\mathrm{Re}\frac{2}{\mathrm{i}r}\oint_{|s|=1}\frac{\mathrm{d}s}{s^2+2s\cosh\phi+1}\end{aligned} \tag{A.12}$$

Introducing the complex variable $s = \mathrm{e}^{\mathrm{i}\varphi}$ we can write (A.12) as complex integral around the unit circle in the complex s-plane. Taking into account that the denominator of (A.12) can be factorized $s^2 + 2\cosh\phi s + 1 = (s + \mathrm{e}^{\phi})(s + \mathrm{e}^{-\phi})$ with zeros $s_{1,2} = -\mathrm{e}^{\pm\phi}$ and $s_1 s_2 = 1$, we can evaluate (A.12) by utilizing the residue theorem where we need only the zero *within* the unit circle $|s| = 1$. Observing that only the residue at $s_1 = -\mathrm{e}^{-\phi}$ (where we can choose $\phi = |\phi|$) is located within the unit circle $|s| = 1$ and therefore contributes to (A.12), we obtain

$$\bar{g}(r,t) = \frac{c\Theta(t)}{2\pi r}\mathrm{Re}\frac{1}{\sinh\phi} \tag{A.13}$$

which is non-zero only if ϕ is real, i.e. for $ct/r > 1$. Thus we arrive at

$$\bar{g}(r,t) = \frac{1}{2\pi}\frac{\Theta\left(t-\frac{r}{c}\right)}{\sqrt{t^2-\left(\frac{r}{c}\right)^2}} \tag{A.14}$$

Expression (A.14) describes the physical propagation of an outgoing singular circular plane wave with propagation velocity c. On a circle with radius r around the source point $r = 0$ the wave arrives only when $t = r/c$. For $t > r/c$ Green's function $\bar{g}(r,t)$ is non-vanishing. For $t < r/c$ the circular wave is not yet arrived at the circle with radius r, therefore the Green's function then is vanishing.

By utilizing (A.3) we obtain for $\bar{h}$ the expression

$$\bar{h}(r,t) = \frac{\Theta(t-\frac{r}{c})}{2\pi}\left\{t\ln\left(\frac{ct}{r}+\sqrt{\frac{c^2t^2}{r^2}-1}\right)-\sqrt{t^2-\frac{r^2}{c^2}}\right\} \tag{A.15}$$

where $\bar{g}(r,t) = (\mathrm{d}^2/\mathrm{d}t^2)\bar{h}(r,t)$. In view of (30) and with (A.14) and (A.15) we can construct the Green's function (34) in the space time domain completely. Furthermore it is a small step to obtain Green's function (A.14) in the space frequency domain. To that end we have to determine $\bar{g}(r,\omega)$ which is defined by

$$\bar{g}(r,\omega) = \int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}\bar{g}(r,t)\,\mathrm{d}t \tag{A.16}$$

Inserting (A.14) into (A.16) leads to

$$\bar{g}(r,\omega) = \frac{1}{2\pi}\int_{\frac{r}{c}}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t}{\sqrt{t^2-\frac{r^2}{c^2}}} \tag{A.17}$$

which can be transformed by putting $ct/r = \cosh\phi$ into

$$\bar{g}(r,\omega+\mathrm{i}\epsilon) = \frac{1}{2\pi}\int_0^{\infty}\mathrm{e}^{\mathrm{i}\frac{(\omega+\mathrm{i}\epsilon)r}{c}\cosh\phi}\,\mathrm{d}\phi \tag{A.18}$$

Furthermore, a definition of the Hankel function of the first kind is (Courant and Hilbert, 1968)

$$H_0^1(z) = \frac{2}{\pi \mathrm{i}} \int_0^\infty \mathrm{e}^{\mathrm{i}z\cosh\phi}\,\mathrm{d}\phi \tag{A.19}$$

Taking into account (A.19) we obtain for $\bar{g}(r,\omega)$ of (A.18) and by putting $k_0 = (\omega/c) + \mathrm{i}\epsilon$

$$\bar{g}(r,\omega + \mathrm{i}\epsilon) = \frac{\mathrm{i}}{4} H_0^1(k_0 r) \tag{A.20}$$

and taking into account definition of $\bar{h}(r,t)$, Eq. (A.2) we observe

$$\bar{h}(r,\omega + \mathrm{i}\epsilon) = -\frac{\bar{g}(r,\omega + \mathrm{i}\epsilon)}{\omega^2} = -\frac{\mathrm{i}}{4\omega^2} H_0^1\left(\frac{(\omega + \mathrm{i}\epsilon)r}{c}\right) \tag{A.21}$$

Expressions (A.20), (A.21) and (A.14), (A.15) represent the Green's functions of Eqs. (A.1) and (A.2) in the space frequency- and the causal ones in the space-time domain, respectively. From these results we obtain by taking into account Fourier transform (30) Green's function (34) itself and its *causal* space-time representation of the components of the quasiplane dynamic Green's function and arrive at

$$G_{ik}(r,t) = \left\{ \frac{\theta_{ik}}{C_{66}^0} \bar{g}_2(r,t) + \frac{1}{\rho_0} \frac{\partial^2}{\partial x_\mathrm{i} \partial x_k} \left[\bar{h}_1(r,t) - \bar{h}_2(r,t) \right] + \frac{m_i m_k}{C'_{44}} \bar{g}_3(r,t) \right\} \tag{A.22}$$

$$\gamma_i(r,t) = \frac{e_{15}^0}{\eta_{11}^0 C'_{44}} m_i \bar{g}_3(r,t) \tag{A.23}$$

$$g(r,t) = \frac{\delta(t)}{2\pi\eta_{11}^0} \ln r + \frac{(e_{15}^0)^2}{(\eta_{11}^0)^2 C'_{44}} \bar{g}_3(r,t) \tag{A.24}$$

where $\bar{g}_i$ and $\bar{h}_i$ denote functions of the form (A.14) and (A.15) with the wave velocities c_i given by

$$c_1 = \sqrt{\frac{C_{11}^0}{\rho_0}}, \quad c_2 = \sqrt{\frac{C_{66}^0}{\rho_0}}, \quad c_3 = \sqrt{\frac{C'_{44}}{\rho_0}}, \quad C'_{44} = C_{44}^0 + \frac{(e_{15}^0)^2}{\eta_{11}^0} \tag{A.25}$$

Appendix B

In order to develop the Green's function defined in (39) we consider the integral of Eq. (41)

$$G(r,a,k_0) = \int_S \bar{g}(\mathbf{r} - \mathbf{r}', k_0)\,\mathrm{d}^2 r' \tag{B.1}$$

where $k_0 = (\omega/c) + \mathrm{i}\epsilon$, $r = |\mathbf{r}|$ and a denotes the radius of the circular inhomogeneity. $g(r,k_0)$ denotes the dynamic Green's function due to a point source defined by

$$(\Delta + k_0^2)\bar{g}(r,k_0) = -\delta^2(\mathbf{r}) \tag{B.2}$$

where $\Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ is the Laplace operator and $k_0 = (\omega/c) + \mathrm{i}\epsilon$. The imaginary part $\epsilon \to 0+$ again guarantees that the problem and all corresponding integrals become well defined. From Eq. (A.20) we have

$$\bar{g}(r,k_0) = \frac{\mathrm{i}}{4} H_0^1(k_0 r) \tag{B.3}$$

G defined by (B.1) corresponds to the Green's function of a circular source distribution represented by the inhomogeneity S. Correspondingly we observe the property

$$\begin{aligned}(\Delta + k_0^2)G(r, a, k_0) &= \int_S (\Delta + k_0^2)\bar{g}(\mathbf{r} - \mathbf{r}', k_0)\,\mathrm{d}^2r' \\ &= -\int_S \delta^2(\mathbf{r} - \mathbf{r}')\,\mathrm{d}^2r' \\ &= -\Theta_\mathrm{S}(\mathbf{r}) \\ &= -\Theta(a - r)\end{aligned} \tag{B.4}$$

where Θ_S is the characteristic function of the inhomogeneity expressed by the Heaviside step function $\Theta(a-r)$ being 1 if $r < a$ and 0 for $r > a$.

We can express via Fourier transformation integral (B.1) in the form

$$G(r, a, k_0) = \frac{1}{(2\pi)^2}\int \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}r}}{k^2 - k_0^2}\,\mathrm{d}^2k \int_S \mathrm{e}^{-\mathrm{i}\mathbf{k}r'}\,\mathrm{d}^2r' \tag{B.5}$$

leading to

$$G(r, a, k_0) = \frac{a}{2\pi}\int \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}r}}{k^2 - k_0^2}\frac{J_1(ka)}{k}\,\mathrm{d}^2k = a\int_0^\infty \frac{J_0(kr)J_1(ka)}{k^2 - k_0^2}\,\mathrm{d}k \tag{B.6}$$

It is convenient to derive this integral first in the k-t domain. With

$$g(k, t) = \frac{1}{(2\pi)}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{k^2 - k_0^2}\,\mathrm{d}\omega = c\Theta(t)\frac{\sin(ckt)}{k} \tag{B.7}$$

and by using $-(1/k)(\mathrm{d}J_0(ka)/\mathrm{d}a) = J_1(ka)$ together with

$$\frac{1}{k^2} = \int_0^\infty \mathrm{e}^{-ky}y\,\mathrm{d}y \tag{B.8}$$

we obtain

$$\begin{aligned}G(r, a, t) &= -ac\Theta(t)\frac{\mathrm{d}}{\mathrm{d}a}\int_0^\infty J_0(kr)J_0(ka)\frac{\sin(ckt)}{k^2}\,\mathrm{d}k \\ &= -\frac{ca\Theta(t)}{(2\pi)^2}\frac{\mathrm{d}}{\mathrm{d}a}\int_0^{2\pi}\int_0^{2\pi}\mathrm{d}\varphi_1\mathrm{d}\varphi_2\mathrm{Im}\int_0^\infty y\,\mathrm{d}y\int_0^\infty \mathrm{e}^{-k(y-\mathrm{i}(ct+r\cos\varphi_1+a\cos\varphi_2+\mathrm{i}\epsilon))}\,\mathrm{d}k \\ &= -\frac{ca\Theta(t)}{(2\pi)^2}\frac{\mathrm{d}}{\mathrm{d}a}\int_0^{2\pi}\int_0^{2\pi}\mathrm{d}\varphi_1\mathrm{d}\varphi_2\,\mathrm{Im}\int_0^\infty \frac{y\,\mathrm{d}y}{(y - \mathrm{i}(ct + r\cos\varphi_1 + a\cos\varphi_2 + \mathrm{i}\epsilon))} \\ &= \frac{ca\Theta(t)}{(2\pi)^2}\frac{\mathrm{d}}{\mathrm{d}a}\mathrm{Re}\int_0^{2\pi}\int_0^{2\pi}\mathrm{d}\varphi_1\mathrm{d}\varphi_2(ct + r\cos\varphi_1 + a\cos\varphi_2)\ln(ct + r\cos\varphi_1 + a\cos\varphi_2 + \mathrm{i}\epsilon) \\ &= \frac{ca\Theta(t)}{(2\pi)^2}\mathrm{Re}\int_0^{2\pi}\int_0^{2\pi}\mathrm{d}\varphi_1\mathrm{d}\varphi_2\cos\varphi_2\ln(ct + r\cos\varphi_1 + a\cos\varphi_2 + \mathrm{i}\epsilon)\end{aligned} \tag{B.9}$$

where all expressions are considered in the limiting case $\epsilon \to 0+$. The last relation is a very compact formulation of (B.1) in the r–t-domain. Remember that this integral can also be represented by

$$G(r, a, t) = \int_S \bar{g}(|\mathbf{r} - \mathbf{r}'|, t)\,\mathrm{d}^2r' \tag{B.10}$$

where $\bar{g}(r,t)$ is given in (A.14). Corresponding to (B.4) $G(r,a,t)$ fulfills the equation

$$\left(\frac{1}{c^2}\frac{d^2}{dt^2} - \Delta\right)G(r,a,t) = \Theta_s(\mathbf{r})\delta(t) = \Theta(a-r)\delta(t) \tag{B.11}$$

For evaluation of (B.1) we first consider (B.9)

$$G(r,a,t) = \frac{ca\Theta(t)}{(2\pi)^2}\int_0^{2\pi}\mathcal{G}(r,a,t,\varphi)\,d\varphi \tag{B.12}$$

We now have to distinguish two cases, namely $r < a$ and $r > a$.

Case I: $r < a$, the spacepoint $\mathbf{r}$ is located *inside* the inhomogeneity.

Then we have to consider an integral of the form

$$\mathcal{G}(r,a,t,\varphi) = \mathrm{Re}\int_0^{2\pi}\cos\alpha\ln(\cos\alpha + A(\varphi,r,t,a))\,d\alpha \tag{B.13}$$

where we put $A = (r/a)\cos\varphi + (ct/a) + i\epsilon$. We have to distinguish $A = \cosh\phi(\varphi,r,t,a) > 1$, (i.e. ϕ real, then we may put $\epsilon = 0$) and $|A| < 1$, i.e. $\phi = i(\psi - i\delta)$ thus $A = \cos(\psi - i\delta)$, where $\delta \to 0+$ is an infinitesimal real constant which is needed for regularization of integral (B.13) to be well defined. It is important to note that for $|A| < 1$, we have to take the real part of (B.13). To evaluate (B.13) it is convenient to expand the logarithm in the following series [11]

$$\begin{aligned}\ln(\cos\alpha + \cosh\phi)\cos\alpha &= \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})\left[\ln\frac{e^{\phi}}{2} + \ln(1 + e^{-\phi}e^{i\alpha}) + \ln(1 + e^{-\phi}e^{-i\alpha})\right]\\ &= \frac{1}{2}(e^{i\alpha} + e^{-i\alpha})\left[\ln\frac{e^{\phi}}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{-n\phi}}{n}(e^{in\alpha} + e^{-in\alpha})\right]\end{aligned} \tag{B.14}$$

Only the terms of zero order in $e^{i\alpha}$ contribute to (B.13), namely

$$\begin{aligned}\mathcal{G}(r,a,t,\varphi) &= 2\pi\mathrm{Re}\{e^{-\phi(r,a,t,\varphi)}\}\\ &= 2\pi e^{-\phi(r,a,t,\varphi)} = 2\pi\left(A - \sqrt{(A^2-1)}\right),\ \text{if } A = \cosh\phi > 1(\phi > 0)\\ &= 2\pi\mathrm{Re}\{e^{-i\psi(r,a,t,\varphi)}\} = 2\pi\cos\psi = 2\pi A,\ \text{if } A = \cos\psi,\ \phi = i(\psi - i\delta)\end{aligned} \tag{B.15}$$

where for $A = 1$ both equations coincide and yield $\mathcal{G} = 2\pi$. If $|A| < 1$ integral (B.12) or equivalently (B.10) and (B.9) assumes the form

$$\begin{aligned}G(r,a,t) &= \frac{ca\Theta(t)}{(2\pi)}\int_0^{2\pi}\left(\frac{r}{a}\cos\varphi + \frac{ct}{a}\right)d\varphi\\ G(r,a,t) &= c^2t\Theta(t), \quad \text{if } t < (a-r)/c,\ (r < a)\end{aligned} \tag{B.16}$$

In view of (B.9) and (B.10) the result (B.16) is worth while to observe: For $0 < t < (a-r)/c$ integral (B.9) or equivalently (B.10) depends *linearly* on time, (i.e. when $|A| < 1$ for all φ). Physically the result (B.16) can be interpreted as the superposition effect of waves arriving at $\mathbf{r}$ emitted at $t = 0$ by point sources aligned on circles with increasing radii ct around $\mathbf{r}$. This superposition effect continues until the "first" wave from the boundary of the circle arrives at $t_c = (a-r)/c$. For times larger than t_c, Eq. (B.16) no longer

[11] Without loss of generality we can choose $\mathrm{Re}\,\phi > 0$ to guarantee that the following series is convergent. For $|A| < 1$ this is especially ensured by introducing the regularization parameter $\delta \to 0+$.

holds. Since for $t < (a-r)/c$ no waves from the boundary arrive at **r**, it is to be expected that (B.16) is independent from radius a and of r, since this effect occurs for all space points **r** in the same way for $r < a$. This situation changes only for $t > (a-r)/c$, since then superposition of waves emitted from different boundary points occurs at space point **r**.

Now it is only a small step to derive by using (B.15) and (B.16) integral (B.1) for $r < a$ in the r–ω-domain. To that end we have to calculate

$$\begin{aligned} G(r,a,k_0) &= \int_{-\infty}^{\infty} G(r,a,t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t \\ &= c^2\int_0^{\infty} \mathrm{e}^{-(\epsilon-\mathrm{i}\omega)t} t\Theta(t)\,\mathrm{d}t - \frac{a^2}{2\pi}\int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{\mathrm{i}(\omega+\mathrm{i}\epsilon)(\frac{a}{c}\cosh\phi - \frac{r}{c}\cos\varphi)}\sinh^2\phi\,\mathrm{d}\phi\,\mathrm{d}\varphi \end{aligned} \tag{B.17}$$

where $\cosh\phi = \frac{ct}{a} + \frac{r}{a}\cos\varphi$. Integrals (B.17) yield

$$G(r,a,k_0) = -\frac{1}{k_0^2} - a^2 J_0(k_0 r)\int_0^{\infty} \mathrm{e}^{\mathrm{i}k_0 a\cosh\phi}\sinh^2\phi\,\mathrm{d}\phi \tag{B.18}$$

Take now into account the definition of Hankel functions (Courant and Hilbert, 1968)

$$H_n^1(k_0 a) = \frac{\mathrm{e}^{-\mathrm{i}\frac{n\pi}{2}}}{\pi\mathrm{i}}\int_0^{\infty} \mathrm{e}^{\mathrm{i}k_0 a\cosh\phi}(\mathrm{e}^{n\phi} + \mathrm{e}^{-n\phi})\,\mathrm{d}\phi \tag{B.19}$$

especially

$$H_1^1(k_0 a) = -\frac{2}{\pi}\int_0^{\infty} \mathrm{e}^{\mathrm{i}k_0 a\cosh\phi}\cosh\phi\,\mathrm{d}\phi \tag{B.20}$$

after partial integration (and by using that k_0 has an infinitesimal imaginary part) we can transform this integral into

$$H_1^1(k_0 a) = \frac{2\mathrm{i}k_0 a}{\pi}\int_0^{\infty} \mathrm{e}^{\mathrm{i}k_0 a\cosh\phi}\sinh^2\phi\,\mathrm{d}\phi \tag{B.21}$$

which is an integral of the type occurring in (B.18), thus we obtain for (B.18) our final result of Eq. (46)

$$G(r,a,k_0) = \frac{1}{k_0^2}\left(\frac{\mathrm{i}\pi}{2}k_0 a J_0(k_0 r) H_1^1(k_0 a) - 1\right) \tag{B.22}$$

which holds for $r < a$, that is **r** is located inside the circular inhomogeneity. We note that it is easily checked that (B.22) indeed fullfills (B.4) for $r < a$.

For the sake of completeness we calculate now (B.1) also for $r > a$:

Case II: $r > a$, the space point **r** is located *outside* the inhomogeneity.

It is now convenient to write (B.12) in the form

$$G(r,a,t) = \frac{ca\Theta(t)}{(2\pi)^2}\int_0^{2\pi} \mathscr{G}(r,a,t,\varphi)\cos\varphi\,\mathrm{d}\varphi \tag{B.23}$$

$$\mathscr{G}(r,a,t,\varphi) = \mathrm{Re}\int_0^{2\pi} \ln(\cos\alpha + A(\varphi,r,t,a))\,\mathrm{d}\alpha \tag{B.24}$$

Note that the cosine cofactor is now integrated in the *first* of both equations, namely already in (B.23). Now we have put $A = \cosh\phi(\varphi,r,t,a) = (a/r)\cos\varphi + (ct/r) + \mathrm{i}\epsilon$. With the same variable substitutions as in (B.15) and expansion (B.14), integral (B.24) yields

$$\begin{aligned}\mathcal{G}(r,a,t,\varphi) &= 2\pi\mathrm{Re}\left\{\ln\left(\frac{\mathrm{e}^{\phi}}{2}\right)\right\}\\ &= 2\pi(\phi(r,a,t,\varphi)-\ln 2),\quad \text{if } A>1,\ (\phi>0)\\ &= 2\pi\mathrm{Re}\left\{\ln\left(\frac{\mathrm{e}^{\mathrm{i}\Psi}}{2}\right)\right\} = 2\pi\mathrm{Re}\{\mathrm{i}\psi-\ln 2\} = -2\pi\ln 2,\quad \text{if } \phi=\mathrm{i}(\psi-\mathrm{i}\delta)\end{aligned} \tag{B.25}$$

We see that in the last case (B.25, third term), because then $\mathcal{G}$ is a φ-independent constant, integral (B.23) is vanishing. Vanishing of (B.23) means physically nothing but the runtime effect described above: Integral (B.10) is vanishing if $A = (a/r)\cos\varphi + (ct/r) < 1$ and this is fulfilled for all φ if $t < (r-a)/c$. Thus we can state

$$G(r,a,t) = 0,\quad \text{if } t < \frac{(r-a)}{c},\ r>a \tag{B.26}$$

This relation is fulfilled as $t < (r-a)/c$ which is the runtime of the wave emitted from the closest boundary point of the inhomogeneity to $\mathbf{r}$ which has the distance $r-a$. By using (B.24) we obtain for (B.23) for arbitrary t the integral

$$G(r,a,t) = \frac{ca}{2\pi}\mathrm{Re}\int_0^{2\pi}\phi(r,a,t,\varphi)\cos\varphi\,\mathrm{d}\varphi \tag{B.27}$$

with $\cosh\phi = A = (a/r)\cos\varphi + (ct/r)$. The real part implicates that (B.27) is only non-vanishing if $\phi > 0$ or equivalently if $A > 1$. From here it is now a small step to derive (B.1) also for $r > a$:

We only have to evaluate

$$G(r,a,k_0) = \frac{ca}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\cos\varphi\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\omega+\mathrm{i}\epsilon)t}\mathrm{Re}\{\phi(r,a,t,\varphi)\}\,\mathrm{d}t \tag{B.28}$$

Because of $\cosh\phi = A = (a/r)\cos\varphi + (ct/r)$ we have $t = (r/c)\cosh\phi - (a/c)\cos\varphi$ and $\sinh\phi\,\mathrm{d}\phi = (c/r)\mathrm{d}t$, integral (B.28) assumes the form

$$G(r,a,k_0) = \frac{ar}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\cos\varphi\,\mathrm{e}^{-\mathrm{i}\frac{a}{c}(\omega+\mathrm{i}\epsilon)\cos\varphi}\int_0^{\infty}\phi\,\mathrm{e}^{\mathrm{i}(\omega+\mathrm{i}\epsilon)\frac{r}{c}\cosh\phi}\sinh\phi\,\mathrm{d}\phi \tag{B.29}$$

where $k_0 = (\omega+\mathrm{i}\epsilon)/c$. Take now in account that

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\cos\varphi\,\mathrm{e}^{-\mathrm{i}ak_0\cos\varphi} = -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\sin\varphi\,\mathrm{e}^{\mathrm{i}k_0a\sin\varphi} = -\mathrm{i}J_1(k_0a) \tag{B.30}$$

and

$$\begin{aligned}\int_0^{\infty}\phi\,\mathrm{e}^{\mathrm{i}k_0r\cosh\phi}\sinh\phi\,\mathrm{d}\phi &= \phi\frac{1}{\mathrm{i}k_0r}\mathrm{e}^{\mathrm{i}k_0r\cosh\phi}\Big|_0^{\infty} - \frac{1}{\mathrm{i}k_0}\int_0^{\infty}\mathrm{e}^{\mathrm{i}k_0r\cosh\phi}\mathrm{d}\phi\\ &= \frac{\mathrm{i}}{k_0r}\int_0^{\infty}\mathrm{e}^{\mathrm{i}k_0r\cosh\phi}\mathrm{d}\phi = -\frac{\pi}{2k_0r}H_0^1(k_0r)\end{aligned} \tag{B.31}$$

where the first term in (B.31, first term) vanishes because of the infinitesimal term $\mathrm{i}\epsilon$ in its exponent. Using (B.30) and (B.31), expression (B.29) finally yields for $r > a$, i.e. $\mathbf{r}$ outside the inhomogeneity

$$\begin{aligned} G(r,a,k_0) &= a\frac{J_1(k_0a)}{k_0}\frac{\pi \mathrm{i}}{2}H_0^1(k_0r) \\ &= 2\pi a\frac{J_1(k_0a)}{k_0}\frac{\mathrm{i}}{4}H_0^1(k_0r) \\ &= F(a,k_0)\bar{g}(r,k_0) \end{aligned} \tag{B.32}$$

where $\bar{g}(r,k_0) = (\mathrm{i}/4)H_0^1(k_0r)$ is the Green's function (B.3) with the Fourier transform $F(a,k_0)$ at $k = k_0$ of the characteristic function of the inhomogeneity

$$F(a,k_0) = 2\pi a\frac{J_1(k_0a)}{k_0} = \int_S \mathrm{e}^{\mathrm{i}k_0\mathbf{n}(\varphi)\cdot\mathbf{r}'}\mathrm{d}^2r' \tag{B.33}$$

Note that $F(a,k_0)$ tends to πa^2 when $a \to 0$, then $G(r,a,k_0)/\pi a^2$ of Eq. (B.32) corresponds to the Green's function of a point source at $r = 0$ (A.20) as it was to be expected. At this result the required property (B.4) is immediately checked: Because of

$$(\Delta + k_0^2)g(r,k_0) = -\delta^2(\mathbf{r}) \tag{B.34}$$

we obtain from (B.32)

$$\begin{aligned} (\Delta + k_0^2)G(r,a,k_0) &= F(a,k_0)(\Delta + k_0^2)g(r,k_0) \\ &= -F(a,k_0)\delta^2(\mathbf{r}) = 0, \quad \text{since } r > a \end{aligned} \tag{B.35}$$

which corresponds to the required relation (B.4) for $r > a$, that is, $\mathbf{r}$ is located outside the inhomogeneity.

Expressions (B.22) and (B.32) represent the closed form solution of the Green's function (B.1) defined by Eq. (B.4) corresponding to a source distribution which is represented by a circular inhomogeneity.

Appendix C

Here we derive integral (40) which has the form

$$I(r,a) = \int_{S(a)} \mathrm{d}^2r' \ln|\mathbf{r} - \mathbf{r}'| \tag{C.1}$$

and is performed over a circular inhomogeneity with radius a. Let us first transform $I(r,a)$ into an integral over the boundary of the circle. To this end we put $g = \ln|\mathbf{r} - \mathbf{r}'|$ and $f = \frac{1}{4}|\mathbf{r} - \mathbf{r}'|^2$.

Then we have

$$g\Delta_{r'}f - f\Delta_{r'}g = \nabla'(g\nabla'f - f\nabla'g) = \ln|\mathbf{r} - \mathbf{r}'| \tag{C.2}$$

Inserting this expression instead of $\ln|\mathbf{r} - \mathbf{r}'|$ into integral (C.1) yields

$$I(r,a) = \frac{1}{4}\int_{\partial S(a)} \{2\ln|\mathbf{r} - a\mathbf{n}(\varphi)| - 1\}\{a\mathbf{n}(\varphi) - \mathbf{r}\}\mathbf{n}(\varphi)\,\mathrm{dO}(\varphi) \tag{C.3}$$

where $\mathbf{n}$ denotes the outer normal of the circle $\partial S(a)$. We can write for this integral the expression $\mathrm{dO}(\varphi) = a\mathrm{d}\varphi$

$$I(r,a) = \frac{a}{4}\int_0^{2\pi} \mathrm{d}\varphi(a - r\cos\varphi)\big(\ln(a^2 + r^2 - 2ar\cos\varphi) - 1\big) \tag{C.4}$$

or

$$I(r,a) = -\frac{\pi a^2}{2} + \frac{a}{4}\int_0^{2\pi} \mathrm{d}\varphi(a - r\cos\varphi)\ln(a^2 + r^2 - 2ar\cos\varphi) = -\frac{\pi a^2}{2} + \frac{a}{4}\mathscr{I}(r,a) \tag{C.5}$$

We now evaluate the remaining integral $\mathscr{I}(r,a)$ with the help of the residue theorem. To that end let us consider the expression

$$r^2 + a^2 - 2ar\cos\varphi = a^2\left(1 - \frac{r}{a}\mathrm{e}^{\mathrm{i}\varphi}\right)\left(1 - \frac{r}{a}\mathrm{e}^{-\mathrm{i}\varphi}\right) \tag{C.6}$$

thus

$$\ln\left(a^2 + r^2 - 2ar\cos\varphi\right) = 2\ln a + \ln\left(1 - \frac{r}{a}\mathrm{e}^{\mathrm{i}\varphi}\right) + \ln\left(1 - \frac{r}{a}\mathrm{e}^{-\mathrm{i}\varphi}\right) \tag{C.7}$$

and expand the logarithm for $|x| < 1$ into a convergent series corresponding to

$$\ln(1 - x) = -\sum_{n=1}^{\infty}\frac{x^n}{n} \tag{C.8}$$

thus we can write for $r < a$ the expansion

$$\ln(r^2 + a^2 - 2ar\cos\varphi) = 2\ln a - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{a}\right)^n\left(\mathrm{e}^{n\mathrm{i}\varphi} + \mathrm{e}^{-n\mathrm{i}\varphi}\right) \tag{C.9}$$

and

$$a - r\cos\varphi = \frac{a}{2}\left(2 - \frac{r}{a}\left(\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{e}^{-\mathrm{i}\varphi}\right)\right) \tag{C.10}$$

Now introduce the complex variable $s = \mathrm{e}^{\mathrm{i}\varphi}$ and transform $\mathscr{I}(r,a)$ into an integral over the unit circle $|s| = 1$ and take into account that $\mathrm{d}\varphi = \mathrm{d}s/\mathrm{i}s$. To evaluate $\mathscr{I}(r,a)$ we have to determine the residue of the integrand, that is we have to collect all terms containing s^{-1} of the expression

$$\frac{a}{2\mathrm{i}}\left(\frac{2}{s} - \frac{r}{a}\left(1 + \frac{1}{s^2}\right)\right)\left(2\ln a - \frac{r}{a}\left(s + \frac{1}{s}\right)\right) - \frac{1}{2}\left(\frac{r^2}{a^2}\left(s^2 + \frac{1}{s^2}\right) + \cdots\right) \tag{C.11}$$

and obtain by applying the residue theorem

$$\mathscr{I}(r,a) = 2\pi\mathrm{i}\sum\mathrm{Res} = 2\pi a\left(2\ln a + \frac{r^2}{a^2}\right) \tag{C.12}$$

Thus we obtain for (C.4)

$$I(r,a) = \frac{\pi}{2}\left(2a^2\ln(a) - a^2 + r^2\right) \tag{C.13}$$

which holds for $r < a$, that is the space point $\mathbf{r}$ is located *inside* the inhomogeneity and corresponds to (40).

For the sake of completeness we give (40) also for the remaining case $r > a$. The same choice of variables and a corresponding expansion now with respect to a/r of (C.1) finally yields

$$I(r,a) = \pi a^2\ln(r) \tag{C.14}$$

for $r > a$, that is $\mathbf{r}$ is located outside the inhomogeneity. In view of (C.13) and (C.14) we observe furthermore the necessary condition

$$\Delta_r I(r,a) = 2\pi\Theta(a - r) \tag{C.15}$$

which obviously fulfills (40) when we take into account that $\Delta_r\ln(|\mathbf{r} - \mathbf{r}'|) = 2\pi\delta^2(\mathbf{r} - \mathbf{r}')$. Note that expressions (C.13) and (C.14) correspond to the *static* Green's function defined by Eq. (C.15) due to a source distribution represented by a circular inhomogeneity.

Appendix D

Here we give a derivation of the used farfield asymptotics of Eq. (95) based on the method of stationary points. To that end let us consider the integral

$$F(\lambda) = \int_a^b f(x)\exp\{i\lambda S(x)\}dx \tag{D.1}$$

where we consider the case of large (positive) argument $\lambda \gg 1$. To that end we assume that $S(x)$ has *within* $[a, b]$ at $a < x_0 < b$ a stationary point $S(x_0)$ where $(d/dx)S(x_0) = S'(x_0) = 0$ and $(d^2/dx^2)S(x_0) = S''(x_0) \neq 0$. We furthermore assume that $f(x)$ can be expanded into a Taylor series around x_0 that is convergent for $a \leqslant x \leqslant b$ according to

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \cdots \tag{D.2}$$

and according to our assumptions the series of $S(x)$ has the form

$$S(x) = S(x_0) + \frac{1}{2!}S''(x_0)(x - x_0)^2 + \cdots \tag{D.3}$$

To evaluate $F(\lambda)$ from (D.1) it is convenient to introduce the new variable $u = \lambda(x - x_0)$. Thus (D.1) can be written as

$$F(\lambda) = \int_{\lambda(a-x_0)}^{\lambda(b-x_0)} \left[f(x_0) + f'(x_0)\frac{u}{\lambda} + \frac{1}{2!}f''(x_0)\frac{u^2}{\lambda^2} + \cdots\right] \exp i\left\{\lambda S(x_0) + \frac{u^2}{2\lambda}S''(x_0) + \cdots\right\}\frac{du}{\lambda} \tag{D.4}$$

We now consider the limiting case $\lambda \gg 1$. Then the integration limits of (D.4) tend to $\pm\infty$, respectively and we have

$$F(\lambda) = \frac{\exp i\{\lambda S(x_0)\}}{\lambda}\int_{-\infty}^{+\infty}\left[f(x_0) + f'(x_0)\frac{u}{\lambda} + \frac{1}{2!}f''(x_0)\frac{u^2}{\lambda^2} + \cdots\right]\exp i\left\{\frac{u^2}{2\lambda}S''(x_0) + \cdots\right\}du \tag{D.5}$$

The integral containing the linear term $f'(x_0)(u/\lambda)$ vanishes, since u is an odd function whereas the exponential is an even function of u (at least when λ tends to infinity where the higher powers as u^2/λ^2 can be neglected). Thus we have as dominant terms in (D.5) the expressions

$$F(\lambda) = \frac{\exp i\{\lambda S(x_0)\}}{\lambda}\left(f(x_0)\int_{-\infty}^{+\infty}\exp i\left\{\frac{u^2}{2\lambda}S''(x_0)\right\}du + \frac{1}{2\lambda^2}f''(x_0)\int_{-\infty}^{+\infty}\exp i\left\{\frac{u^2}{2\lambda}S''(x_0)\right\}u^2\,du\right) \tag{D.6}$$

To evaluate (D.6) we have to evaluate integrals of the form $(a = S''(x_0)/(2\lambda))$

$$I(a) = \int_{-\infty}^{\infty}\exp\{iau^2\}\,du \tag{D.7}$$

and

$$J(a) = \int_{-\infty}^{\infty}u^2\exp\{iau^2\}\,du \tag{D.8}$$

where we observe that $J(a) = (1/i)(d/da)I(a)$. Despite $I(a)$ looks trivial we carefully evaluate it here (because of its complex argument). Consider therefore $I^2(a)$ after performing the φ-integration

$$I^2(a) = 2\pi\int_0^{\infty}u\exp\{iau^2\}\,du \tag{D.9}$$

which can be transformed into with $v = u^2$

$$I^2(a) = \lim_{\epsilon \to 0} \pi \int_0^\infty \exp\{-(\epsilon - \mathrm{i}a)v\}\,\mathrm{d}v \tag{D.10}$$

where we have introduced an infinitesimal positive parameter $\epsilon \to 0+$ to define this integral and arrive at

$$I^2(a) = \lim_{\epsilon \to 0} \frac{\pi}{(\epsilon - \mathrm{i}a)} = \pi^2 \delta(a) + \frac{\mathrm{i}\pi}{a} \tag{D.11}$$

because of $a = S''(x_0)/(2\lambda) \neq 0$ we can omit the δ function and furthermore use that $a = |a|\mathrm{sign}(a)$, thus

$$I^2(a) = \frac{\mathrm{i}\ \mathrm{sign}(a)\pi}{|a|} \tag{D.12}$$

and

$$I(a) = \exp\left\{\mathrm{i}\ \mathrm{sign}(a)\frac{\pi}{4}\right\}\sqrt{\frac{\pi}{|a|}} = \sqrt{\frac{\mathrm{i}\pi}{a}} \tag{D.13}$$

and $J(a) = (1/\mathrm{i})(\mathrm{d}/\mathrm{d}a)I(a)$ from (D.8) is then obtained by

$$J(a) = \frac{\mathrm{i}}{2}\exp\left\{\mathrm{i}\ \mathrm{sign}(a)\frac{\pi}{4}\right\}\sqrt{\frac{\pi}{|a|^3}} = \frac{-1}{2\mathrm{i}}\sqrt{\frac{\mathrm{i}\pi}{a^3}} \tag{D.14}$$

and we have from (Eq. (D.5))

$$F(\lambda) = \frac{\exp \mathrm{i}\{\lambda S(x_0)\}}{\lambda}\left(f(x_0)I(a) + \frac{1}{2\lambda^2}f''(x_0)J(a)\right) \tag{D.15}$$

to arrive at by using (D.13) and (D.14) with $\mathrm{sign}(a) = \mathrm{sign}[S''(x_0)]$

$$F(\lambda) = \sqrt{\frac{2\pi}{\lambda|S''(x_0)|}}\exp \mathrm{i}\left\{\lambda S(x_0) + \frac{\pi}{4}\mathrm{sign}[S''(x_0)]\right\}\left(f(x_0) + \frac{\mathrm{i}f''(x_0)}{2\lambda|S''(x_0)|}\right) \tag{D.16}$$

In the case when $S(x)$ has m stationary points x_v ($v = 1, \ldots, m$) with $a < x_v < b$ we obtain instead of (D.16)

$$F(\lambda) = \sum_{v=1}^{m}\sqrt{\frac{2\pi}{\lambda|S''(x_v)|}}\exp \mathrm{i}\left\{\lambda S(x_v) + \frac{\pi}{4}\mathrm{sign}[S''(x_v)]\right\}\left(f(x_v) + \frac{\mathrm{i}f''(x_v)}{2\lambda|S''(x_v)|}\right) \tag{D.17}$$

corresponding to Eq. (95).

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